THE UNIVERSITY OF ALBERTA

INSTABILITY IN LAMINAR BOUNDARY-LAYER FREE CONVECTION

by

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A THESIS

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ABSTRACT

This analysis presents a theoretical boundary-layer formulation which is valid for steady, two-dimensional and axisymmetric free-convection flow of any Newtonian fluid along surfaces having smooth, continuous temperature distributions. In addition, the stability of these flows is analysed theoretically using the small disturbance theory. The linearized forms of the disturbance equations are presented for two forms of small disturbances: two-dimensional disturbances of the form of Tollmien-Schlichting waves, and three-dimensional disturbances of the form of Taylor-Goertler roll vortices.

Numerical solutions are presented for steady, constant-property, boundary-layer free-convection flow of air along long, inclined, isothermal, plane surfaces. These steady- or mean-flow solutions are subsequently used in the stability analysis to obtain numerical solutions for points on the neutral stability curves for both forms of the disturbances.

The mean-flow solutions are found to be in good agreement with previous theoretical and experimental results, where these results are comparable. It is concluded from the mean-flow results that over the range of surface inclinations from 0° to 120° the most significant change in the profiles occurs within the range from 0° to 60° with only very small changes over the range from 60° to 120°.

The results of the stability analysis are in good agreement with previous theoretical results, but not with previous experimental results. However, the difference between the theoretical and experimental results is attributed to an amplification process. The stability of the flow is strongly influenced by the surface inclination over the range from 30° to

120° with the most significant change occurring at about 85°. For angles below 85°, the results reveal that the Taylor-Goertler roll-vortex disturbances determine the stability of the flow, while above 85°, the Tollmien-Schlichting wave disturbances determine the stability.

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Nomenclature

A	$\kappa_c Y_c$
a	A function of x used in the surface-temperature distribution
B	normalized frequency
b	thickness of material between surface and heaters
C_c	characteristic specific heat at constant pressure
C_p	specific heat at constant pressure
C_1	coefficient used in transformation from x, y to η, ξ coordinate system
C_2, C_3	coefficients in linear combination of linearly independent solutions
C	$[\lambda^2 - \frac{iB}{\sigma} Ra_L^{\frac{1}{5}}(1+\omega^2)]^{\frac{1}{2}}$
c	normalized specific heat at constant pressure
D	$[\lambda^2 - iB\ Ra_L^{\frac{1}{5}}(1+\omega^2)]^{\frac{1}{2}}$
d	a function of ω used in Falkner-Skan transformation
E	error in the Newton-Raphson process
F	transformed normalized stream function in η,ξ coordinate system
f	transformed normalized stream function in ζ,ξ coordinate system

$\vec{G}$	body force vector
g	gravitational acceleration
H	a function of ω used in order-of-magnitude analysis
h	a function of ω used in the Lefevre transformation
i	$(-1)^{\frac{1}{2}}$
J	$\frac{Y_c}{R_c}$
j	normalized radius of axisymmetric body
K	normalized coefficient of thermal conductivity
k	coefficient of thermal conductivity
L	a reference length in the X direction representing the maximum distance from the leading edge at which the flow is examined.
M	normalized coefficient of viscosity
N_1, N_2, N_3, N_4	exponents in transformation from x,y to η,ξ coordinate system
Nu_X	local Nusselt number = $\frac{-X}{\Theta_w} \frac{\partial\Theta}{\partial Y}\Big\|_{Y=0}$
n	exponent in transformed surface-temperature distribution
$O[\ ]$	order of magnitude of [ ]
Os	Ostrach number = $\frac{\beta_g X_c}{C_c}$

P	pressure
p	normalized pressure
Q	function used in the Newton-Raphson process
q	ratio of $\frac{\kappa}{\kappa_c}$
R	radius of axisymmetric body
Ra_{X_c}	Rayleigh number based on $X_c = \frac{\beta g \Theta_c X_c{}^3 \sigma}{\nu^2}$
r	disturbance density amplitude function
S	general function in description of finite-difference formulae
s	disturbance temperature amplitude function
T	temperature
t	time
U	velocity component in X direction
u	normalized velocity component in x direction
$\vec{V}$	velocity vector
V	velocity component in Y direction
v	normalized velocity component in y direction
W	velocity component in Z direction

w	normalized velocity component in z direction
X,Y,Z	spatial coordinates
x,y,z	normalized coordinates
∇	vector operator

Greek Symbols

α	angle between surface tangent and the horizontal
β	coefficient of thermal expansion at constant pressure
$\Gamma_1,\Gamma_2,\Gamma_3$	eigenvalues in the mean flow
γ	normalized density
Δ	disturbance specific heat amplitude function
$\delta_1,\delta_2,\delta_3$	disturbance velocity amplitude functions
ε	$\beta\bar{\Theta}_c$
ζ	transformed coordinate introduced by a Lefevre transformation
η	transformed coordinate introduced by a Falkner-Skan transformation
Θ	temperature difference = $T - T_\infty$
θ	normalized temperature difference
κ	curvature in the X direction of the surface Y = 0

Λ	disturbance viscosity amplitude function
λ	wavenumber in x* direction
$\mu,\acute{\mu}$	first (or shear) coefficient of viscosity, coefficient of bulk viscosity
ν	kinematic viscosity
Ξ_1,Ξ_2,Ξ_3	functions used in the Lefevre transformation
ξ	transformed coordinate
Π	transformed pressure in η,ξ coordinate system
π	transformed pressure in ζ,ξ coordinate system
ρ	density of the fluid
Σ	$C_1 Ra_L^{-\omega^2} \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5-4\omega^2}} q\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}$, or summation
σ	Prandtl number = $\frac{\mu C_c}{k}$
τ	normalized time
Υ	disturbance thermal conductivity amplitude function
Φ	transformed temperature in η,ξ coordinate system
ϕ	disturbance stream function amplitude function
χ	transformed temperature in ζ,ξ coordinate system
ψ	normalized stream function

Ω	wavenumber in z direction
ω	normalized angle

Subscripts

c	characteristic value; conjugate when used with vector notation, i.e. $(\nabla\vec{V})_c$
d	departure
i	imaginary part of a complex number
o	hydrostatic
L	evaluated at $X = L$
P	constant pressure
r	real part of a complex number
w	wall
ζ,η,ξ	partial differentiation with respect to ζ,η,ξ
∞	ambient, far from surface

Superscripts

$'$	associated with bulk viscosity
o	reference to position chosen as leading edge
*	reference to disturbance quantity
$-$	reference to mean-flow quantity

CHAPTER I

INTRODUCTION

1.1 Previous Analyses of Steady, Laminar, Free-Convection Boundary-Layer Flows.

One of the earliest analyses of free-convection flows dates back to the theoretical work of Lorenz [1]* in 1881. He considered the problem of a free-convection flow along a heated vertical plate, and in his analysis, Lorenz assumed that the stream lines and the isotherms are parallel to the surface of the plate. However, it is now known that these assumptions do not agree with experimental observations.

In 1930, Schmidt and Beckmann [2] suggested that the boundary-layer assumptions might be applied to the equations for at least some free-convection flows. Using these assumptions, they obtained solutions for a free-convection flow of air over a heated vertical plate. Schmidt and Beckmann also performed an experimental analysis for the same conditions. They used a quartz-fibre anemometer to measure the velocities and manganese-constantan thermocouples to measure the temperatures across the boundary layer.

Pohlhausen [1] appears to have been the first to suggest a similarity transformation of the boundary-layer equations. He determined the unknown conditions at the surface from the experimental results of

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* The numbers in [ ] denote the references.

Schmidt and Beckmann and obtained a series solution for the transformed boundary-layer equations. However, his results are restricted to the flow conditions applicable to Schmidt and Beckmann's experiments. To overcome this restriction Saunders [1] suggested using polynomial approximations to solve the single fifth-order equation which results from a combination of the transformed momentum and energy equations.

Ostrach [3] presented the first "exact" numerical solutions of the transformed free-convection boundary-layer equations for the flow about an isothermal vertical plate. He obtained the velocity and temperature profiles and the heat transfer for a range of Prandtl numbers from 0.01 to 1000. Ostrach found that his theoretical solutions agreed well with the experimental results of Schmidt and Beckmann. Ostrach's work was followed by many other numerical solutions encompassing a wide range of conditions. A few of the analyses dealing with vertical surfaces are briefly discussed below.

Sparrow and Gregg [4] considered a free-convection boundary-layer flow along a vertical plate with a uniform heat flux from the surface. They determined a similarity formulation of the equations and presented solutions for a range of Prandtl numbers from 0.1 to 100. In a later study, Sparrow and Gregg [5] considered non-isothermal vertical surfaces which admit to similarity transformations. They presented solutions for several examples of each of two families of power-law temperature distributions. Sparrow and Gregg [6] also presented similarity solutions for several cases of variable-property flows.

Each of the above analyses is concerned with a free-convection flow along a surface having a continuous temperature distribution.

Shetz and Eichhorn [7] performed experiments using a vertical plate with a single step discontinuity in the surface temperature. They used a Mach Zehnder interferometer to obtain the temperature and heat transfer data for air. They also performed experiments in water using Tellurium dye to observe the flow. Hayday, Bowlus and McGraw [8] analysed the problem theoretically using the concept of local similarity in conjunction with a transformation similar to that used by Falkner and Skan [9]: this type of transformation will henceforth be called a Falkner-Skan transformation.

Combined free- and forced-convection boundary-layer flows along vertical surfaces have also been investigated. Sparrow, Eichhorn and Gregg [10] obtained a similarity transformation for this problem and presented velocity, temperature and heat transfer results for several examples of aiding and opposing flows. Szewczyk [11] approached the problem somewhat differently. He set out to determine the effect of free convection on forced-convection flows and vice versa. For each case he determined the first three terms of a power series expansion where the first term was the forced-convection flow in the first case and the free-convection flow in the second case. Kubair and Pei [12] again used a similarity formulation to determine the solutions for combined free- and forced-convection flows of several non-Newtonian fluids.

Eichhorn [13] analysed the effect of mass transfer on a free-convection boundary-layer flow along a vertical, isothermal plate. He found that for certain power-law distributions of mass transfer the problem could be solved in terms of similarity solutions. Eichhorn considered a range of conditions which covered both suction and blowing, and his results show that mass transfer has a very strong effect on the heat transfer but only a small effect on the skin friction.

Most analyses of free-convection boundary-layer flows assume that the fluid outside the boundary layer is isothermal. An exception is the work of Cheesewright [14] which deals with an isothermal vertical surface in a non-isothermal environment. Cheesewright found that there are certain temperature distributions of the surrounding fluid for which similarity solutions can be obtained. For several examples of these temperature distributions, Cheesewright presented the velocity, temperature and heat transfer results in graphical form and also tabulated the similarity solutions for two examples.

Each of the above analyses has dealt with free-convection boundary-layer flows along vertical plates, but another important surface geometry is a vertical cylinder. In analysing vertical-cylinder problems, Millsaps and Pohlhausen [15] found that the flow along an isothermal vertical cylinder does not admit to a similarity solution. However, they did find that for a linear surface-temperature distribution the equations admitted to similarity solutions. They presented several numerical solutions for the velocity and temperature profiles and compared their heat transfer results with results obtained by applying integral methods. The comparison revealed that integral methods using parabolic approximations of the velocity and temperature profiles give heat transfer results close to the exact numerical solutions.

Kuiken [16] also investigated free-convection boundary-layer flows along vertical cylinders as part of his analysis of the effect of a small radius of curvature on the flow. He considered the flow past thin vertical cylinders and slender vertical cones. He applied a Falkner-Skan transformation and used a series expansion to obtain solutions for several non-linear surface-temperature distributions.

Braun, Ostrach and Heighway [17] extended the analysis of free-convection flows along vertical bodies to a wider class of two-dimensional and axisymmetric body shapes for which similarity solutions could be obtained. Using an integral method, they determined the growth of the boundary-layer thickness and the heat transfer for several specific body shapes for isothermal surfaces. Non-isothermal surface-temperature distributions for which similarity solutions could be obtained were also discussed.

One of the earliest theoretical investigations of free-convection boundary-layer flows along a horizontal surface was the analysis of Stewartson [18]. He presented a similarity formulation of the equations and also obtained a solution for an isothermal surface. However, in his analysis of the equations, Stewartson concluded that a boundary-layer flow on an upward-facing, heated, semi-infinite horizontal surface was impossible but such a flow was possible on a downward-facing, heated, semi-infinite horizontal surface. These conclusions were the reverse of what Stewartson should have concluded, and this error was pointed out by Gill, Zeh and del Casal [19] who showed that the sign of one term in Stewartson's analysis was incorrect.

Rotem and Claassen [20] further examined boundary-layer flows along horizontal surfaces by considering similarity formulations for power-law surface-temperature distributions and determining the limits of the exponents for which solutions can be obtained. They specifically discussed the isothermal and uniform-heat-flux temperature distributions, but they presented numerical solutions for the isothermal case only. They also considered solutions which are independent of the Prandtl number provided that the Prandtl number is tending to the appropriate limit. In addition to their theoretical analysis, Rotem and Claassen performed some experiments

using semi-focusing colour-Schlieren photography to obtain temperature data and what appeared to be reasonable evidence for the existence of a boundary-layer flow on a horizontal semi-infinite surface. They used a variety of plate sizes and covered a range of Rayleigh numbers from 0 to 40,000. In a second paper, Rotem and Claassen [21] extended the above theoretical analysis to axially symmetric flows.

The vertical and horizontal surfaces are rather special cases of a more general class of inclined surfaces. It appears that the earliest investigation of free-convection flows along an inclined surface was that of Rich [22]. Rich suggested that the boundary-layer equations for a flow along a vertical surface could be modified for inclined surfaces simply by modifying the buoyancy term. He made no attempt to include the effects of pressure variations, and he made no attempt to solve the equations. Instead, Rich examined the flows experimentally using a Mach Zehnder interferometer to determine the heat transfer from flat plates heated isothermally and inclined at angles up to forty degrees from the vertical.

Levy [23] was perhaps the first investigator to present some theoretical solutions to the problem of free-convection boundary-layer flows along inclined surfaces. He was concerned with the applicability of integral methods to free-convection problems, and one of the examples he referred to was the inclined isothermal plate. He concluded that the accuracy of the heat transfer results was good, and he found his results for the inclined plates were in agreement with the experimental results of Rich [22]. Michiyoshi [24] also applied an integral method to a free-convection flow along an inclined surface. He considered the heat transfer from an isothermal, infinitely-wide, thin plate of elliptical cross section.

In his analysis, Michiyoshi appears to have neglected the effect of the longitudinal pressure gradient and, therefore, his results cannot be expected to be accurate for inclinations near the horizontal.

Kierkus [25] used another approach to the solution of the boundary-layer equations for free-convection flows along inclined surfaces, namely a perturbation analysis. Using the similarity solution for an isothermal vertical plate as the zeroth-order approximation and using a small parameter relating the surface inclination and the Grashof number as the perturbation parameter, Kierkus determined the first-order approximations for angles up to forty-five degrees on either side of the vertical. To check the validity of his analysis, Kierkus also experimentally determined the velocity, temperature and heat transfer results. He found good agreement between theory and experiment and concluded that his analysis was valid.

Another more recent experimental analysis of free-convection flows along inclined surfaces was conducted by Hassan and Mohamed [26] who were primarily interested in the heat transfer results. To obtain these results, Hassan and Mohamed used Boelter-Schmidt heat flux meters. Their experiments were performed in air for isothermal surfaces and the results cover a range of surface inclinations from 0° to 180°. Wherever comparisons were possible, their results were in good agreement with other experimental results [2, 22, 25] and with the available theoretical results [3, 25].

1.2 Previous Analyses of the Stability of Free-Convection Boundary-Layer Flows

The above discussion indicates that the interest in free-convection flows is considerable. The analyses considered above were all

concerned with steady, laminar flows; however, another important aspect of free-convection flows is the stability of the flow. Some of the analyses of stability and transition to turbulent flows are discussed in this section.

Saunders [1, 27] was one of the earliest to study transition, but he pointed out that in 1922 Griffiths and Davis had noted the transition to turbulent flow on a heated vertical cylinder. Saunders [27] performed his experiments on a heated vertical plate placed inside a pressurized tank. He was primarily interested in determining the effect of pressure on the heat transfer and his study of the transition to a turbulent flow was only a minor aspect of his work. However, this study did reveal that the position of the transition is very sensitive to draughts. Since his apparatus was relatively free from draughts, Saunders obtained a higher value for the transition Grashof numbers than that obtained by Griffiths and Davis.

Eckert and Soehnghen [28] were among the earliest to perform experiments with the primary purpose of studying the stability and the transition to turbulence of free-convection boundary-layer flows. They used a Mach Zehnder interferometer to study the flow along an isothermal, vertical plate. They observed the formation and amplification of a wave disturbance which was initially two-dimensional and purely sinusoidal. However, as the disturbance amplified further, the wave form became more complex and eventually gave way to a three-dimensional disturbance. On the basis of their observations, Eckert and Soehnghen concluded that the flow initially becomes unstable due to some small wave disturbance having a particular wavenumber.

The first theoretical attempt to predict the stability of free-convection boundary-layer flows was made by Plapp [29]. Using the small disturbance theory, Plapp derived the disturbance equations including the coupling between the disturbance momentum and energy equations. He accounted for the effect of surface inclination and demonstrated how to account for variable properties. However, since he lacked the necessary computer facilities, Plapp could only obtain approximate solutions of the uncoupled equations for the case of constant-property flows. In particular, he obtained the neutral stability curve for the uncoupled equations for a free-convection flow over an isothermal, vertical plate.

Birch [30] performed another experimental analysis of a free-convection flow about an isothermal, vertical plate. He studied the stability of the flow by introducing small disturbances at various distances from the leading edge of the plate. He introduced these disturbances by passing an electrical pulse through a wire placed in the flow, and he observed the flow with a Mach Zehnder interferometer.

Another experimental technique was utilized by Eckert, Hartnett and Irvine [31] for the purpose of detecting three-dimensional effects in the transition process. Interferometers cannot detect these effects since the density of the observed fluid is integrated along the light beam. Therefore, Eckert, Hartnett and Irvine used smoke threads to obtain information to supplement the interferometric studies of free-convection flow about an isothermal, vertical plate. They also used this technique to determine the effect on the stability of the flow of placing an obstacle in the flow at certain positions along the surface. They found that the critical Grashof number decreased as the distance of the obstacle from the leading edge increased.

A theoretical and experimental analysis of the stability of a free-convection boundary-layer flow of water around an isothermal, vertical plate was presented by Szewczyk [32]. Szewczyk used expansions about the critical points to obtain solutions to the uncoupled disturbance equations. For his experimental work, he used a thermocouple probe to measure the temperature, and he used dye-injection to observe the flow field.

Nachtsheim [33] was the first to obtain a numerical solution of the coupled disturbance equations. Previous investigators had always assumed that the effect of coupling was small, but Nachtsheim showed that this effect was very significant. He determined both the coupled and uncoupled neutral stability curves for free-convection flows of air and water along an isothermal, vertical plate. The coupled results revealed the existence of two modes of instability: one in which energy is transferred to the disturbance by Reynolds stresses, and one in which energy is transferred to the disturbance by the interaction of the buoyancy forces with the velocity fluctuations.

The disturbance equations for a vertical plate having a uniform heat flux from the surface were derived by Polymeropoulos and Gebhart [34]. Polymeropoulos and Gebhart solved the uncoupled equations and showed how to convert the results for an isothermal plate for comparison with the uniform-heat-flux results. An experimental analysis of this problem was also performed by Polymeropoulos and Gebhart [35]. They introduced artificial disturbances by means of a vibrating ribbon placed in the flow, and they determined the approximate position of the neutral stability curve by carefully observing the disturbances with a Mach Zehnder interferometer. The results were in good agreement with Nachtsheim's theoretical results [33] and definitely revealed the importance of coupling.

Knowles and Gebhart [36] solved the disturbance equations for the case of a free-convection boundary-layer flow along a vertical surface having a uniform heat flux. They showed that a thermal-capacity coupling exists between the fluid and the surface and that this coupling has a first-order effect on the solution. They solved the equations for several values of the thermal capacity for air, and the results showed the importance of this coupling. The results were also in good agreement with the experimental results of Polymeropoulos and Gebhart [35]. In addition, Knowles and Gebhart chose a zero-thermal-capacity surface and obtained solutions for several Prandtl numbers.

In another analysis of the stability of a free-convection boundary-layer flow along a uniform-heat-flux vertical plate, Dring and Gebhart [37] considered the spatial amplification characteristics of the disturbance as it moved downstream. They determined the constant-amplification-rate contours from which the relative amplification was determined. Then using the amplitude of the neutrally stable disturbances as unity, Dring and Gebhart presented the amplitude ratio contours. Their results revealed that low-frequency, long-wavelength disturbances begin to amplify first but higher-frequency, shorter-wavelength disturbances amplify much faster. To check these theoretical predictions, Dring and Gebhart [38] conducted an experimental analysis. They used a hot-wire anemometer to obtain the amplitude and phase profiles of the disturbance velocity and to measure the spatial amplification rate as a function of the frequency. To measure the amplitude and phase profiles of the disturbance temperature, Dring and Gebhart used an interferometer. The relative spatial amplification of the disturbance temperature was determined from interferometric moiré patterns. The experimental results were found to be in good agreement

with the theoretical results, and they verified that the disturbances which are amplified fastest have much higher frequencies than the disturbances which begin to amplify first.

Hieber and Gebhart [39] also considered the stability of free-convection boundary-layer flows along a uniform-heat-flux vertical plate. Using a simplified numerical procedure, they were able to extend the analysis to a much larger range of Grashof numbers. For a Prandtl number of 0.733 as the Grashof number tends to infinity, Hieber and Gebhart found that the effect of temperature coupling vanished more rapidly than the effect of viscosity. They also noted that the upper branch of the neutral stability curve was oscillatory but tending to a non-zero asymptote typical of an inviscid instability. For higher Prandtl numbers, the two instability modes merged to form a loop in the neutral stability curve. As the Prandtl number tended to infinity, the temperature coupling was dominant in the instability. For small Prandtl numbers, there was only one mode of instability with the temperature coupling effect being negligible or highly destabilizing for large or small thermal-capacity walls, respectively. They also presented empirical correlations between the small disturbance theory and the regions in which the flow first becomes significantly oscillatory and in which the flow first departs significantly from a laminar flow.

Hieber and Gebhart [40] further examined the above problem in an attempt to determine the physical effects associated with the two instability modes as the Prandtl number tends to infinity. They used asymptotic expansions of both the mean-flow equations and the disturbance-flow equations to accomplish their goal. From their analysis, Hieber and Gebhart noted that while the boundary-layer theory predicts the formation

of an inner or thermal layer and an outer or viscous layer, the stability theory combined with the empirical correlation with the experimental results predicts that the flow will be turbulent before the two-layer structure can fully develop. Hieber and Gebhart also converted the results to the isothermal case.

All of the above analyses have dealt with vertical surfaces, but some experimental work has been done on the stability of free-convection flows along inclined surfaces. Tritton [41] used a fibre anemometer to study the stability and transition of a free-convection flow along an isothermal plate inclined at angles up to fifty degrees on either side of the vertical. He noted that the buoyancy effect stabilizes the flow for downward-facing surfaces and destabilizes the flow for upward-facing surfaces. The experimental results of Lock, Gort and Pond [42] clearly showed this effect. A Schlieren apparatus and thermocouples were used by Lock, Gort and Pond to obtain a plot of the critical Rayleigh number versus the surface inclination.

Lloyd and Sparrow [43] later confirmed the results of Lock, Gort and Pond; but in addition, they revealed a fundamental change in the mechanism of the instability as the surface inclination passes through a range between 14° and 17° from the vertical for upward-facing surfaces. They found that for angles less than 14° the instability was due to wave disturbances but for angles greater than 17° the instability took the form of longitudinal roll vortices. To study the stability of the flow, Lloyd and Sparrow used the same technique used by Sparrow and Husar [44] in a study of the formation of roll vortices, namely an electrochemical flow visualization technique.

1.3 Present Analysis

The above discussions of steady or mean, laminar, free-convection boundary-layer flows indicates that the theoretical solutions for vertical (90°) and horizontal (0°) surfaces are well established under a wide range of conditions. For other inclinations, solutions are available only by approximate integral methods or perturbation analyses under limited conditions. It is felt that the profiles obtained by integral methods may not be sufficiently accurate for the stability analysis, and the perturbation analysis has a limited range of application. Therefore, one purpose of the present analysis is to attempt to find a boundary-layer formulation of the equations which is valid for two-dimensional and axisymmetric flows of any Newtonian fluid along surfaces having smooth, continuous temperature distributions. (A finite number of finite discontinuities in the surface temperature might be considered by a method used by Hayday, Bowlus and McGraw [8].) Before attempting the boundary-layer formulation, it is also necessary to establish a normalization procedure and an order-of-magnitude analysis which are valid under the conditions stated. A procedure for solving these boundary-layer flows must also be found. In particular, numerical solutions are sought for constant-property, free-convection boundary-layer flows of air along long, inclined, isothermal, plane surfaces. The resulting temperature, pressure, velocity and heat transfer data obtained from this analysis will be compared with available theoretical and experimental results.

A second purpose of this analysis is to study the stability of free-convection boundary-layer flows subjected to two forms of small disturbances: a two-dimensional wave disturbance, and a set of longitudinal roll-vortex disturbances. For the somewhat general mean-flow

conditions stated above, an attempt is made to establish the linearized forms of the disturbance equations based on the small disturbance theory. As in the case of the mean-flow equations, a normalization procedure and an order-of-magnitude analysis have to be established such that they are valid under the conditions stated for the mean flow. Then procedures for solving these equations must be chosen. Solutions are then sought for the stability of constant-property, free-convection boundary-layer flows of air along long, inclined, isothermal, plane surfaces subjected to both forms of disturbances. In particular, the neutral stability curves are to be determined, and from these curves the critical Rayleigh numbers will be determined. The critical Rayleigh numbers for both disturbance forms will be compared to determine the ranges of surface inclinations over which each disturbance form governs the stability of the flow. Finally, the critical Rayleigh numbers will be compared with the available theoretical and experimental data.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Steady-Flow Equations

Steady, two-dimensional or axisymmetric free-convection flow of a Newtonian fluid about a heated surface or in a buoyant jet is considered. (The flow about any cooled surface inclined at 180° to an identical heated surface can be described by the same equations and boundary conditions provided that the property variations of the fluid are skew-symmetric about the ambient temperature.) An attempt is made to reduce the Navier-Stokes equations and the energy equation to a set of equations describing a free-convection boundary-layer flow along the surface, or in the jet. The equations are to be applicable for a temperature distribution which varies continuously with position along the surface, or the axis of the jet, or which has a finite number of finite discontinuities.

2.1-1 Governing Equations

For the present analysis, thermal radiation effects are assumed to be negligible and the above flow is assumed to be described by the following vector equations [45]:

conservation of mass:

$$\nabla.(\rho\vec{V}) = 0,$$

conservation of momentum:

$$\rho\vec{V}.\nabla\vec{V} = -\nabla P + \nabla[(\acute{\mu}-\frac{2}{3}\mu)\nabla.\vec{V}] + (\nabla\mu).[(\nabla\vec{V}) + (\nabla\vec{V})_c] + \mu[\nabla^2\vec{V} + \nabla(\nabla.\vec{V})] + \rho\vec{G}, \qquad (2.1\text{-}1)$$

conservation of energy:

$$\rho C_p \vec{V}.\nabla T + \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \vec{V}.\nabla P = \nabla.(k\nabla T) + (\acute{\mu}-\frac{2}{3}\mu)(\nabla.\vec{V})^2$$

$$+ \mu\nabla.\{[(\nabla\vec{V}) + (\nabla\vec{V})_c].\vec{V}\}-\mu\vec{V}.[\nabla^2\vec{V} + \nabla(\nabla.\vec{V})]$$

In addition to these equations, an equation of state and relations for the property variations are required.

Consider equations 2.1-1 applied to a region such as the one illustrated in figure 1. It is assumed that the body force field is due to gravitational effects alone. It is also assumed that the pressure can be expressed as a sum of the hydrostatic pressure, measured for the case of no flow, and a departure from this hydrostatic pressure, that is:

$$P = P_o + P_d \; .$$

Similarly, the density is expressed as a sum of the density of the undisturbed fluid far from the surface and a departure, that is:

$$\rho = \rho_\infty + \rho_d \; .$$

The pressure, P_o, and the density, ρ_∞, are related as follows:

$$0 = \frac{-1}{(1+\kappa Y)} \frac{\partial P_o}{\partial X} - \rho_\infty g \sin\alpha$$

and $$0 = -\frac{\partial P_o}{\partial Y} - \rho_\infty g \cos\alpha,$$

where $\kappa = \kappa(X)$ is the curvature in the X direction of the surface Y = 0 for any Z.

The set of equations 2.1-1 can now be written as:

$$\rho \frac{\partial U}{\partial X} + U \frac{\partial \rho}{\partial X} + (1+\kappa Y)\left[\rho \frac{\partial V}{\partial Y} + V \frac{\partial \rho}{\partial Y}\right] + \kappa\rho V + \frac{\rho}{(R\pm Y\sin\alpha)}\left[\pm VY \sin\alpha + U\left(\frac{dR}{dX}\pm Y\cos\alpha\frac{d\alpha}{dX}\right)\right]=0$$

$$\frac{\rho U}{(1+\kappa Y)}\frac{\partial U}{\partial X} + \frac{\rho V\partial U}{\partial Y} + \frac{\kappa\rho UV}{(1+\kappa Y)} = \frac{-1}{(1+\kappa Y)}\frac{\partial P_d}{\partial X} + \frac{(\acute{\mu}+4/3\mu)}{(1+\kappa Y)^2}\{\frac{\partial^2 U}{\partial X^2} + (1+\kappa Y)\frac{\partial^2 V}{\partial X\partial Y} + \frac{\kappa\partial V}{\partial X}$$

$$+ \frac{Vd\kappa}{dX} + \frac{1}{(R\pm Y\sin\alpha)}[\frac{Ud^2R}{dX^2} \mp UY\sin\alpha(\frac{d\alpha}{dX})^2 \pm UY\cos\alpha\frac{d^2\alpha}{dX^2} + \frac{\partial U}{\partial X}(\frac{dR}{dX}$$

$$\pm Y\cos\alpha\frac{d\alpha}{dX}) \pm (1+\kappa Y)V\cos\alpha\frac{d\alpha}{dX} \pm (1+\kappa Y)\frac{\partial V}{\partial X}\sin\alpha] - \frac{1}{(R\pm Y\sin\alpha)^2}[U(\frac{dR}{dX}$$

$$\pm Y\cos\alpha\frac{d\alpha}{dX})^2 \pm V\sin\alpha(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})]\} - \frac{(\acute{\mu}+4/3\mu)Y}{(1+\kappa Y)^3}\frac{d\kappa}{dX}[\frac{\partial U}{\partial X} + \kappa V$$

$$+ \frac{U}{(R\pm Y\sin\alpha)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})] + \frac{1}{(1+\kappa Y)^2}\frac{\partial(\acute{\mu}-2/3\mu)}{\partial X}\{\frac{\partial U}{\partial X} + \kappa V$$

$$+ (1+\kappa Y)\frac{\partial V}{\partial Y} + \frac{1}{(R\pm Y\sin\alpha)}[U(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm (1+\kappa Y)V\sin\alpha]\}$$

$$+ \frac{2}{(1+\kappa Y)^2}\frac{\partial\mu}{\partial X}(\frac{\partial U}{\partial X} + \kappa V) + \frac{\partial\mu}{\partial Y}[\frac{\partial U}{\partial Y} + \frac{1}{(1+\kappa Y)}(\frac{\partial V}{\partial X} - \kappa U)] + \mu\{\frac{\partial^2 U}{\partial Y^2}$$

$$- \frac{1}{(1+\kappa Y)}[\frac{\partial^2 V}{\partial X\partial Y} - \frac{\kappa\partial U}{\partial Y} \pm \frac{\sin\alpha}{(R\pm Y\sin\alpha)}(\frac{\partial V}{\partial X} - (1+\kappa Y)\frac{\partial U}{\partial Y} - \kappa U)]$$

$$+ \frac{\kappa}{(1+\kappa Y)^2}(\frac{\partial V}{\partial X} - \kappa U)\} - \rho_d g\sin\alpha \qquad (2.1\text{-}2)$$

$$\frac{\rho U}{(1+\kappa Y)}\frac{\partial V}{\partial X} + \frac{\rho V\partial V}{\partial Y} - \frac{\rho U^2\kappa}{(1+\kappa Y)} = \frac{-\partial P_d}{\partial Y} + (\acute{\mu}+4/3\mu)\{\frac{\partial^2 V}{\partial Y^2} \pm \frac{\sin\alpha}{(R\pm Y\sin\alpha)}[\frac{\partial V}{\partial Y}$$

$$\mp \frac{V\sin\alpha}{(R\pm Y\sin\alpha)}] + \frac{1}{(1+\kappa Y)} [\frac{\partial^2 U}{\partial X \partial Y} + \frac{\kappa \partial V}{\partial Y} + \frac{1}{(R\pm Y\sin\alpha)} (\frac{\partial U}{\partial Y} \frac{dR}{dX} \pm \frac{Y\cos\alpha \partial U}{\partial Y} \frac{d\alpha}{dX}$$

$$\pm U\cos\alpha \frac{d\alpha}{dX} \mp \frac{U\sin\alpha}{(R\pm Y\sin\alpha)} (\frac{dR}{dX} \pm Y\cos\alpha \frac{d\alpha}{dX}))] - \frac{\kappa}{(1+\kappa Y)^2} [\frac{\partial U}{\partial X}$$

$$+ \frac{U}{(R\pm Y\sin\alpha)} (\frac{dR}{dX} \pm Y\cos\alpha \frac{d\alpha}{dX}) + \kappa V]\} + \frac{\partial(\acute{\mu}-2/3\mu)}{\partial Y} [\frac{\partial V}{\partial Y} \pm \frac{V\sin\alpha}{(R\pm Y\sin\alpha)}$$

$$+ \frac{1}{(1+\kappa Y)} \frac{\partial U}{\partial X} + \kappa V + \frac{U}{(R\pm Y\sin\alpha)} (\frac{dR}{dX} \pm Y\cos\alpha \frac{d\alpha}{dX})] + \frac{1}{(1+\kappa Y)} \frac{\partial \mu}{\partial X} [\frac{\partial U}{\partial Y}$$

$$+ \frac{1}{(1+\kappa Y)} (\frac{\partial V}{\partial X} - \kappa U)] + \frac{2\partial\mu}{\partial Y} \frac{\partial V}{\partial Y} - \frac{\mu}{(1+\kappa Y)} \{\frac{\partial^2 U}{\partial X \partial Y} + \frac{1}{(R\pm Y\sin\alpha)} \frac{\partial U}{\partial Y} (\frac{dR}{dX}$$

$$\pm Y\cos\alpha \frac{d\alpha}{dX}) - \frac{1}{(1+\kappa Y)} [\frac{\partial^2 V}{\partial X^2} - \frac{U d\kappa}{dX} - \frac{\kappa \partial U}{\partial X} + \frac{1}{(R\pm Y\sin\alpha)} (\frac{dR}{dX}$$

$$\pm Y\cos\alpha \frac{d\alpha}{dX})(\frac{\partial V}{\partial X} - \kappa U)] + \frac{Y}{(1+\kappa Y)^2} \frac{d\kappa}{dX} (\frac{\partial V}{\partial X} - \kappa U)\} - \rho_d g \cos\alpha$$

$$\frac{\rho C_p U}{(1+\kappa Y)} \frac{\partial \Theta}{\partial X} + \frac{\rho C_p V \partial \Theta}{\partial Y} + \frac{T}{\rho} (\frac{\partial \rho}{\partial T})_P [\frac{U}{(1+\kappa Y)} \frac{\partial P}{\partial X} + \frac{V \partial P}{\partial Y}] = \frac{1}{(1+\kappa Y)^2} [\frac{k \partial^2 \Theta}{\partial X^2}$$

$$+ \frac{k}{(R\pm Y\sin\alpha)} \frac{\partial \Theta}{\partial X} (\frac{dR}{dX} \pm Y\cos\alpha \frac{d\alpha}{dX}) + \frac{\partial k}{\partial X} \frac{\partial \Theta}{\partial X}] + \frac{k \partial^2 \Theta}{\partial Y^2} + \frac{\partial k}{\partial Y} \frac{\partial \Theta}{\partial Y}$$

$$\pm \frac{k\sin\alpha}{(R\pm Y\sin\alpha)} \frac{\partial \Theta}{\partial Y} + \frac{k\kappa}{(1+\kappa Y)} \frac{\partial \Theta}{\partial Y} - \frac{kY}{(1+\kappa Y)^3} \frac{d\kappa}{dX} \frac{\partial \Theta}{\partial X} + (\acute{\mu}+4/3\mu)\{\frac{1}{(1+\kappa Y)^2} [(\frac{\partial U}{\partial X})^2$$

$$+ \kappa^2V^2 + 2\kappa V\frac{\partial U}{\partial X} + \frac{U^2}{(R\pm Y\sin\alpha)^2}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})^2] + (\frac{\partial V}{\partial Y})^2$$

$$+ \frac{V^2\sin^2\alpha}{(R\pm Y\sin\alpha)^2} \pm \frac{2UV\sin\alpha}{(1+\kappa Y)(R\pm Y\sin\alpha)^2}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\} + \frac{2(\acute{\mu}-2/3\mu)}{(R\pm Y\sin\alpha)}\{$$

$$\pm \sin\alpha\frac{\partial V}{\partial Y} + \frac{1}{(1+\kappa Y)}[(R\pm Y\sin\alpha)(\frac{\partial U}{\partial X}\frac{\partial V}{\partial Y} + \kappa V\frac{\partial V}{\partial Y}) \pm \kappa V^2\sin\alpha \pm V\sin\alpha\frac{\partial U}{\partial X}$$

$$+ U\frac{\partial V}{\partial Y}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})] + \frac{U}{(1+\kappa Y)^2}(\frac{\partial U}{\partial X} + \kappa V)(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\}$$

$$+ \mu\{(\frac{\partial U}{\partial Y})^2 + \frac{2}{(1+\kappa Y)}\frac{\partial U}{\partial Y}(\frac{\partial V}{\partial X} - \kappa U) + \frac{1}{(1+\kappa Y)^2}[(\frac{\partial V}{\partial X})^2 + \kappa^2U^2 - 2\kappa U\frac{\partial V}{\partial X}]\}$$

where $\Theta = T - T_\infty$

$\alpha = \alpha(X)$

and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ implies an upward-facing surface.

The above set of equations is not complete until a sufficient number of boundary conditions has been established. However, because of the semi-infinite nature of the region, the complete set of boundary conditions is not evident. This fact combined with the complexity of the equations makes it desirable to seek some simplification to the equations with a possibility of obtaining a boundary-layer formulation. Such a formulation would eliminate the necessity for specifying those boundary

conditions which are not evident.

2.1-2 Boundary-Layer Equations

In an attempt to simplify the equations, the equations are normalized and an order-of-magnitude analysis is performed. This procedure is given in detail in appendix A. The final result of this procedure is a set of equations of the boundary-layer type which may be expressed in dimensionless form as follows:

$$\frac{\gamma\partial u}{\partial x} + \frac{u\partial\gamma}{\partial x} + \frac{\gamma}{(j\pm Jy\sin\alpha)}\left[\pm vJ\sin\alpha + u\left(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx}\right)\right] + (1+Aqy)\left(\frac{\gamma\partial v}{\partial y} + \frac{v\partial\gamma}{\partial y}\right)$$

$$+ Aq\gamma v = 0$$

$$\gamma Ra_L^{\frac{1}{5}(4\omega^2-1)}\left[\frac{u}{(1+Aqy)}\frac{\partial u}{\partial x} + \frac{v\partial u}{\partial y} + \frac{Aquv}{(1+Aqy)}\right] = \frac{-\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{\partial p_d}{\partial x}$$

$$+ \sigma Ra_L^{\frac{1}{5}(4\omega^2-1)}\left[\frac{M\partial^2 u}{\partial y^2} + \frac{\partial M}{\partial y}\frac{\partial u}{\partial y} + \frac{AqM}{(1+Aqy)}\frac{\partial u}{\partial y} \pm \frac{MJ\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial u}{\partial y}\right.$$

$$\left. - \frac{Aqu}{(1+Aqy)}\frac{\partial M}{\partial y} - \frac{A^2Mq^2u}{(1+Aqy)^2} \pm \frac{MAq}{(1+Aqy)}\frac{Ju\sin\alpha}{(j\pm Jy\sin\alpha)}\right] - \frac{\sigma\gamma_d}{\beta\Theta_c}\sin\alpha$$

$$+ 0\left[\sigma Ra_L^{\frac{1}{5}(2\omega^2-3)},\ \sigma A^2 Ra_L^{\frac{1}{5}(2\omega^2-3)}\right] \tag{2.1-3}$$

$$- Ra_L^{\omega^2} \frac{Aq\gamma u^2}{(1+Aqy)} = -\frac{\sigma \partial p_d}{\partial y} - \frac{\sigma \gamma_d}{\beta \Theta_c} \cos\alpha + 0\ [Ra_L^{\frac{1}{5}(3\omega^2-2)}, \sigma Ra_L^{\frac{1}{5}(3\omega^2-2)},$$

$$A^2 \sigma Ra_L^{\frac{1}{5}(3\omega^2-2)}]$$

$$\gamma c[\frac{(1+a)}{(1+Aqy)} \frac{u\partial\theta}{\partial x} + \frac{\theta u}{(1+Aqy)} \frac{da}{dx} + \frac{(1+a)v\partial\theta}{\partial y}] = (1+a)[\frac{K\partial^2\theta}{\partial y^2} + \frac{\partial K}{\partial y}\frac{\partial\theta}{\partial y} + \frac{AKq}{(1+Aqy)}\frac{\partial\theta}{\partial y}$$

$$\pm \frac{KJ\sin\alpha}{(j \pm Jy\sin\alpha)} \frac{\partial\theta}{\partial y}] + 0\ [Os\ Ra_L^{-\frac{1}{5}(1+\omega^2)}, Ra_L^{-\frac{2}{5}(1+\omega^2)}, A\ Ra_L^{-\frac{2}{5}(1+\omega^2)},$$

$$Os\ Ra_L^{\frac{1}{5}(4\omega^2-1)}, A^2 Os\ Ra_L^{\frac{1}{5}(4\omega^2-1)}]$$

2.1-3 Boundary Conditions

Having posed the problem in terms of a boundary layer formulation, it is necessary to establish the applicable boundary conditions. To arrive at these conditions, it is assumed that the presence of the heated surface does not affect the fluid in the region preceding X = 0 or in the region far from the surface. Furthermore, it is assumed that there is no flow parallel to the surface at X = 0. Although experiments indicate that these conditions are not exact, there is no detailed information known *a priori* about the flow outside the boundary layer to facilitate the establishment of more realistic boundary conditions. However, even if it were possible to establish more exact boundary conditions, it must be noted that the

boundary-layer equations 2.1-3 are not valid near X = 0. Thus by applying these equations near X = 0, the solutions obtained will be in error, but it is assumed that this error will propagate only a short distance into the region where the boundary-layer approximations are valid.

The other boundary conditions state that there is no flow across the solid surface, there is no slip of the fluid in contact with the surface, and the fluid at the surface attains the temperature of the surface. Therefore, the relevant nondimensional boundary conditions are:

$$
\begin{aligned}
x = 0,\ y>0: \quad & \gamma = 1 \\
& c = 1 \\
& u = P_d = \theta = 0 \\
x>0,\ y = 0: \quad & \theta = \theta_w(x) \\
& u = v = 0 \\
\text{All } x,\ y\to\infty: \quad & \gamma = 1 \\
& M = 1 \\
& c = 1 \\
& K = 1 \\
& u = P_d = \theta = 0
\end{aligned}
\qquad (2.1\text{-}4)
$$

The condition $\theta = \theta_w(x)$ for $x>0$ and $y = 0$ implies that the surface temperature distribution is known. If the heat flux from the surface were known, this condition would be replaced by a condition on $\frac{\partial\theta}{\partial y}$ at the surface.

2.2 Constant-Property Flow

2.2-1 Governing Equations

In general, the fluid properties are functions of temperature

and pressure. However, the departure from hydrostatic pressure resulting from a free-convection flow is usually of a magnitude which has little or no effect on the fluid properties. If temperature variations are small throughout the fluid, the property variations resulting from the temperature variations can frequently be ignored.

Consider all fluid properties constant with the exception of the density. The density is also considered to be constant in all terms except the buoyancy terms, in which the Boussinesq approximation is used, that is:

$$\rho = \rho_\infty[1-\beta(T-T_\infty)].$$

Using the expression for the density as given in section 2.1-1, it follows that:

$$\rho_d = -\rho_\infty\beta\Theta$$

The set of equations 2.1-3 now becomes:

$$\frac{\partial u}{\partial x} + \frac{u}{(j \pm Jy\sin\alpha)}\left(\frac{dj}{dx} \pm Jy\cos\alpha\,\frac{d\alpha}{dx}\right) + (1+Aqy)\frac{\partial v}{\partial y} + Aqv = 0$$

$$Ra_L^{\frac{1}{5}(4\omega^2-1)}\left[\frac{u}{(1+Aqy)}\frac{\partial u}{\partial x} + \frac{v\partial u}{\partial y} + \frac{Aquv}{(1+Aqy)}\right] = -\frac{\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{\partial p_d}{\partial x}$$

$$+ \sigma Ra_L^{\frac{1}{5}(4\omega^2-1)}\left[\frac{\partial^2 u}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial u}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial u}{\partial y} - \frac{A^2q^2u}{(1+Aqy)^2}\right.$$

$$\left.\pm \frac{Aq}{(1+Aqy)}\frac{J\,u\,\sin\alpha}{(j\pm Jy\sin\alpha)}\right] + \sigma(1+a)\,\theta\sin\alpha$$

(2.2-1)

$$-Ra_L^{\omega^2}\frac{Aq\,u^2}{(1+Aqy)} = \frac{-\sigma\partial p_d}{\partial y} + \sigma(1+a)\,\theta\cos\alpha$$

$$\frac{(1+a)u}{(1+Aqy)}\left(\frac{\partial\theta}{\partial x}+\frac{\theta da}{dx}\right)+\frac{(1+a)v\partial\theta}{\partial y}=(1+a)\left[\frac{\partial^2\theta}{\partial y^2}+\frac{Aq}{(1+Aqy)}\frac{\partial\theta}{\partial y}\pm\frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\theta}{\partial y}\right]$$

The corresponding boundary conditions are as follows:

$$x = 0,\ y > 0:\quad u = P_d = \theta = 0$$

$$x > 0,\ y = 0:\quad u = v = 0,\ \theta = \theta_w(x) \qquad (2.2\text{-}2)$$

$$\text{All } x,\ y \to \infty:\quad u = P_d = \theta = 0$$

2.2-2 The Falkner-Skan Formulation

In order to proceed further with the analysis of the above equations, consideration must be given to the geometry of the surface. The following analysis is concerned primarily with two-dimensional surfaces. Whenever possible, an attempt is made to outline the procedure for asixymmetric bodies and for buoyant jets.

The equation of conservation of mass, or the continuity equation as it is frequently called, as it appears in equations 2.2-1 or 2.1-3 can be identically satisfied by the introduction of a stream function, ψ, defined as follows:

$$\frac{\partial\psi}{\partial y} = u \text{ and } \frac{\partial\psi}{\partial x} = -(1+Aqy)v, \qquad (2.2\text{-}3)$$

or

$$\frac{\partial\psi}{\partial y} = \gamma u \text{ and } \frac{\partial\psi}{\partial x} = -\gamma(1+Aqy)v, \qquad (2.2\text{-}4)$$

respectively.

In addition to the stream function, the Falkner-Skan transformation [9] is introduced as follows:

$$\psi(x,y) = x^{\frac{n+3}{5-4\omega^2}} F(\eta, \xi),$$

$$P_d(x,y) = x^{\frac{4n+2+8\omega^2}{5-4\omega^2}} \Pi(\eta, \xi), \qquad (2.2\text{-}5)$$

and $\theta(x,y) = x^n \Phi(\eta, \xi)$,

where $\xi = x$,

$$\eta(x,y) = \frac{y}{C_1} x^{\frac{n-2+4\omega^2}{5-4\omega^2}}$$

and C_1 is a constant used in scaling the boundary layer thickness in the η, ξ plane. The derivation of this transformation is given in appendix B. It is also pointed out in appendix B how the transformation can be modified for flows for which it is either impossible or inconvenient to introduce a stream function, such as in the case of axisymmetric flows.

Introducing the transformations given in equations 2.2-3 and 2.2-5 into equations 2.2-1 leads to the following set of equations:

$$C_1 Ra_L^{\frac{1}{5}(4\omega^2-1)} \{(\frac{2n+1+4\omega^2}{5-4\omega^2}) F_\eta^{\ 2} - (\frac{n+3}{5-4\omega^2}) FF_{\eta\eta} + \xi(F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta})$$

$$- C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} [(\frac{n+3}{5-4\omega^2}) FF_\eta + (\frac{n-2+4\omega^2}{5-4\omega^2}) \eta F_\eta^{\ 2} + \xi F_\xi F_\eta]$$

$$/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})\}/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

$$= -\sigma C_1^{\,3} Ra_L^{-\frac{1}{5}(1+\omega^2)} [(\frac{4n+2+8\omega^2}{5-4\omega^2}) \Pi + (\frac{n-2+4\omega^2}{5-4\omega^2}) \eta \Pi_\eta + \xi \Pi_\xi]$$

$$/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}) + \sigma Ra_L^{\frac{1}{5}(4\omega^2-1)} \{F_{\eta\eta\eta} + C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} F_{\eta\eta}$$

$$/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}) - C_1^{\,2}A^2q^2\xi^{\frac{4-2n-8\omega^2}{5-4\omega^2}} F_\eta/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})^2\}$$

$$+ \sigma(1+a)\, C_1^{\,3}\xi^{\frac{(n+3)(1-4\omega^2)}{5-4\omega^2}} \Phi\sin\alpha \tag{2.2-6}$$

$$- Ra_L^{\omega^2} Aq\xi^{\frac{(2-n)(1-4\omega^2)}{5-4\omega^2}} F_\eta^{\,2}/[C_1^{\,2}(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})] = \frac{-\sigma}{C_1}\xi^{\frac{4\omega^2(n+3)}{5-4\omega^2}} \Pi_\eta$$

$$+ \sigma(1+a)\, \Phi\cos\alpha$$

$$C_1\{n\,\Phi\, F_\eta - (\frac{n+3}{5-4\omega^2})\, \Phi_\eta\, F + \xi[\Phi_\xi F_\eta - \Phi_\eta F_\xi + \frac{\Phi F_\eta a_\xi}{(1+a)}]\}/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

$$= \Phi_{\eta\eta} + C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} \Phi_\eta/(1+C_1 Aq\eta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}),$$

where the subscripts ξ and η refer to partial differentiation with respect to ξ and η, respectively.

Before considering the boundary conditions to apply to equations 2.2-6, it is important to note that the function, $a(x)$, is chosen such that $\Theta_w(X) = \Theta_w(L)[1+a(x)]\, x^n$, which leads to a very simple transformed boundary condition.

The boundary conditions now become the following:

$$
\begin{aligned}
&F(0,\xi) = F_\eta(0,\xi) = \Phi(0,\xi) - 1 = 0 \\
&F_\eta(\infty,\xi) = \Phi(\infty,\xi) = \Pi(\infty,\xi) = 0 \\
&F(\eta,0) = F(\eta,\xi^\circ) \\
&F_\eta(\eta,0) = F_\eta(\eta,\xi^\circ) \\
&\Phi(\eta,0) = \Phi(\eta,\xi^\circ) \\
&\Pi(\eta,0) = \Pi(\eta,\xi^\circ)
\end{aligned}
\tag{2.2-7}
$$

where ξ° is some small, but non-zero, value of ξ used in the lateral momentum equation to avoid the singularity which arises in an attempt to solve for Π_η at $\xi = 0$.

Equations 2.2-6 are applicable to a reasonably wide class of problems. However, the primary purpose of this work is to consider a class of problems for which the curvature of the surface is zero, that is $q = 0$. For this particular class for the special cases of $\alpha = 0$ and $\alpha = \pi/2$ (with $a(x) = 0$), equations 2.2-6 are replaceable by ordinary differential equations since the transformations given by equations 2.2-5 can be replaced by similarity transformations. However, since equations 2.2-6 are valid for $a(x) = 0$ for $\alpha = 0$ and $\alpha = \pi/2$, it must follow that all terms involving derivatives with respect to ξ are identically zero. Similarity solutions were first obtained for $\alpha = 0$ and $\alpha = \pi/2$ by Stewartson [18] and Ostrach [3], respectively. Similarity solutions for a class of problems for which the curvature is non-zero were considered by Braun, Ostrach and Heighway [17].

2.2-3 The Lefevre Transformation

Equations 2.2-6 are applicable to fluids having a Prandtl number of the order of unity or higher. It would be most desirable to obtain a set of equations which would apply for any Prandtl number, including the limits of zero and infinity. Lefevre [46] introduced a transformation, hereafter referred to as the Lefevre transformation, which was suitable for all Prandtl numbers including zero and infinity. Using Lefevre's basic idea, the transformation for the present analysis is derived in appendix C and is given by the following:

$$\eta = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5-4\omega^2}} \zeta,$$

$$F(\eta,\xi) = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{-1}{5-4\omega^2}} f(\zeta,\xi),$$

$$\Pi(\eta,\xi) = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5-4\omega^2}} \pi(\zeta,\xi), \tag{2.2-8}$$

and $\quad \Phi(\eta,\xi) = \chi(\zeta,\xi).$

Introducing equations 2.2-8 into equations 2.2-6 and setting

$\Sigma = C_1 A \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5-4\omega^2}} q\, \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}$ leads to:

$$C_1 Ra_L^{\frac{1}{5}(4\omega^2-1)} \{(\frac{2n+1+4\omega^2}{5-4\omega^2})\, f_\zeta^{\,2} - (\frac{n+3}{5-4\omega^2})\, f\, f_{\zeta\zeta} + \xi(f_\zeta\, f_{\xi\zeta} - f_\xi\, f_{\zeta\zeta})$$

$$- \Sigma[(\frac{n+3}{5-4\omega^2})\, f\, f_\zeta + (\frac{n-2+4\omega^2}{5-4\omega^2})\zeta\, f_\zeta^{\,2} + \xi\, f_\xi\, f_\zeta]/(1+\zeta\Sigma)\}/(1+\zeta\Sigma)$$

$$= - C_1{}^3\sigma\left(\frac{1+\sigma}{\sigma}\right)^{\frac{5}{5-4\omega^2}} Ra_L^{-\frac{1}{5}(1+\omega^2)} \left[\left(\frac{4n+2+8\omega^2}{5-4\omega^2}\right)\pi + \left(\frac{n-2+4\omega^2}{5-4\omega^2}\right)\zeta\pi_\zeta\right.$$

$$\left. + \xi\pi_\xi\right]/(1+\zeta\Sigma) + \sigma Ra_L^{\frac{1}{5}(4\omega^2-1)} \{f_{\zeta\zeta\zeta} + \frac{\Sigma}{(1+\zeta\Sigma)} f_{\zeta\zeta} - \frac{\Sigma^2}{(1+\zeta\Sigma)^2} f_\zeta$$

$$+ C_1{}^3\,\sigma(1+a)\left(\frac{1+\sigma}{\sigma}\right)^{\frac{4}{5-4\omega^2}} \xi^{\frac{(n+3)(1-4\omega^2)}{5-4\omega^2}} \chi\sin\alpha \qquad (2.2\text{-}9)$$

$$- q\left(\frac{1+\sigma}{\sigma}\right)^{\frac{-4}{5-4\omega^2}} \xi^{\frac{(2-n)(1-4\omega^2)}{5-4\omega^2}} f_\zeta{}^2/[C_1{}^2(1+\zeta\Sigma)] = -\frac{\sigma}{C_1}\xi^{\frac{4\omega^2(n+3)}{5-4\omega^2}} \pi_\zeta$$

$$+ \sigma(1+a)\chi\cos\alpha$$

$$\frac{C_1}{(1+\zeta\Sigma)}\{n\,\chi\,f_\zeta - \left(\frac{n+3}{5-4\omega^2}\right)\chi_\zeta\,f + \xi[\chi_\xi\,f_\zeta - \chi_\zeta\,f_\xi + \frac{a\xi}{(1+a)}\chi\,f_\zeta]\} = \chi_{\zeta\zeta} + \frac{\Sigma}{(1+\zeta\Sigma)}\chi_\zeta$$

The boundary conditions are:

$$\begin{aligned}
&f(0,\xi) = f_\zeta(0,\xi) = \chi(0,\xi) - 1 = 0\\
&f_\zeta(\infty,\xi) = \chi(\infty,\xi) = \pi(\infty,\xi) = 0\\
&f(\zeta,0) = f(\zeta,\xi^\circ)\\
&f_\zeta(\zeta,0) = f_\zeta(\zeta,\xi^\circ) \qquad\qquad (2.2\text{-}10)\\
&\chi(\zeta,0) = \chi(\zeta,\xi^\circ)\\
&\pi(\zeta,0) = \pi(\zeta,\xi^\circ).
\end{aligned}$$

The set of equations 2.2-9 with the boundary conditions 2.2-10 is only applicable to constant-property flows. However, the Falkner-Skan transformation and the Lefevre transformation are not restricted to such flows. Given some additional relationships to specify the property variations, it would be possible to introduce additional dependent variables into the derivations of these transformations, but the resulting transformation would not be general since they would depend on the specific property-variation relationships. No attempt will be made to give any examples for variable property fluids since the present work is primarily concerned with constant-property fluids.

2.3 Disturbance Equations

The combined hydrodynamic and thermal stability of the steady, free-convection fluid flows described in the preceding section is considered. Using the method of small disturbances, an attempt is made to reduce the time-dependent Navier-Stokes equations and the time-dependent energy equation to a set of disturbance equations which determines the stability of the flow.

2.3-1 Governing Equations

The small disturbance theory assumes that each of the dependent variables is expressible as a sum of a steady-state component and a small disturbance component. In the present analysis, the steady-state (or mean) component is indicated by a bar over the variable and the disturbance component is indicated by an asterisk. In formulating the analysis, the disturbance components are assumed to be three-dimensional and time-dependent. It is also assumed that the disturbance components are sufficiently small to permit the neglect of terms which are quadratic or higher-

order in the disturbance components. Therefore, the governing equations are the three-dimensional forms of equations 2.1-1 with the addition of the unsteady-flow terms. Assuming that the mean components satisfy equations 2.1-2, the resulting set of equations is the following:

$$\frac{\partial\rho^*}{\partial t} + \frac{1}{(1+\kappa Y)}\left[\frac{\bar{\rho}\partial U^*}{\partial X} + \bar{\rho}\kappa V^* + \frac{\rho^*\partial\bar{U}}{\partial X} + \rho^*\kappa\bar{V} + \frac{\bar{U}\partial\rho^*}{\partial X} + \frac{U^*\partial\bar{\rho}}{\partial X} + \frac{1}{(R\pm Y\sin\alpha)}\left(\frac{dR}{dX}\right.\right.$$

$$\left.\left.\pm Y\cos\alpha\frac{d\alpha}{dX}\right)\left(\bar{\rho}U^* + \rho^*\bar{U}\right)\right] + \frac{\bar{\rho}\partial V^*}{\partial Y} + \frac{\rho^*\partial\bar{V}}{\partial Y} + \frac{\bar{V}\partial\rho^*}{\partial Y} + \frac{V^*\partial\bar{\rho}}{\partial Y} \pm \frac{\sin\alpha}{(R\pm Y\sin\alpha)}\left(\bar{\rho}V^*\right.$$

$$\left.+ \rho^*\bar{V}\right) + \frac{R_L}{(R\pm Y\sin\alpha)}\frac{\partial W^*}{\partial Z} = 0$$

$$\frac{\bar{\rho}\partial U^*}{\partial t} + \bar{\rho}\left[\frac{1}{(1+\kappa Y)}\left(\frac{\bar{U}\partial U^*}{\partial X} + \kappa\bar{U}V^* + \frac{U^*\partial\bar{U}}{\partial X} + \kappa U^*\bar{V}\right) + \frac{\bar{V}\partial U^*}{\partial Y} + \frac{V^*\partial\bar{U}}{\partial Y}\right] + \rho^*\left[\frac{\bar{V}\partial\bar{U}}{\partial Y}\right.$$

$$\left.+ \frac{1}{(1+\kappa Y)}\left(\frac{\bar{U}\partial\bar{U}}{\partial X} + \kappa\bar{U}\bar{V}\right)\right] = \frac{-1}{(1+\kappa Y)}\frac{\partial P^*}{\partial X} + \frac{1}{(1+\kappa Y)^2}\frac{\partial}{\partial X}\left(\bar{\mu}' + \frac{4\bar{\mu}}{3}\right)\left(\frac{\partial U^*}{\partial X}\right.$$

$$\left.+ \kappa V^*\right) + \frac{1}{(1+\kappa Y)}\frac{\partial}{\partial X}\left(\bar{\mu}' - \frac{2\bar{\mu}}{3}\right)\left\{\frac{\partial V^*}{\partial Y} + \frac{1}{(R\pm Y\sin\alpha)}\left[\frac{U^*}{(1+\kappa Y)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\right.\right.$$

$$\left.\left.\pm V^*\sin\alpha + R_L\frac{\partial W^*}{\partial Z}\right]\right\} + \frac{(\bar{\mu}'+4/3\bar{\mu})}{(1+\kappa Y)}\left\{\frac{1}{(1+\kappa Y)}\left[\frac{\partial^2 U^*}{\partial X^2} - \frac{Y}{(1+\kappa Y)}\frac{d\kappa}{dX}\frac{\partial U^*}{\partial X}\right.\right.$$

$$\left.+ V^*\frac{d\kappa}{dX} - \frac{\kappa Y V^*}{(1+\kappa Y)}\frac{d\kappa}{dX}\right] + \frac{1}{(R\pm Y\sin\alpha)}\left[\frac{1}{(1+\kappa Y)}\left(U^*\left[\frac{d^2R}{dX^2} \mp Y\cos\alpha\left(\frac{d\alpha}{dX}\right)^2\right.\right.\right.$$

$$\left.\pm Y\cos\alpha\frac{d^2\alpha}{dX^2}\right] + \left[\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right]\left[\frac{\partial U^*}{\partial X} - \frac{U^*Y}{(1+\kappa Y)}\frac{d\kappa}{dX}\right]\right) \pm \frac{V^*\cos\alpha d\alpha}{dX}$$

$$- (\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})[\frac{U^*}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm V^*\sin\alpha]/(R \pm Y\sin\alpha)]\}$$

$$+ \frac{(\bar{\mu}+1/3\bar{\mu})}{(1+\kappa Y)} [\frac{\partial^2 V^*}{\partial Y \partial X} \pm \frac{\sin\alpha}{(R \pm Y\sin\alpha)} \frac{\partial V^*}{\partial X} + \frac{R_L}{(R \pm Y\sin\alpha)} \frac{\partial^2 W^*}{\partial Z \partial X}]$$

$$- \frac{(\bar{\mu}+7/3\bar{\mu})}{(1+\kappa Y)} [\frac{R_L}{(R \pm Y\sin\alpha)^2} (\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \frac{\partial W^*}{\partial Z} - \frac{\kappa}{(1+\kappa Y)} \frac{\partial V^*}{\partial X}]$$

$$+ \frac{1}{(1+\kappa Y)^2} \frac{\partial}{\partial x}(\acute{\mu}^*+\frac{4\mu^*}{3})(\frac{\partial \bar{U}}{\partial X} + \kappa\bar{V}) + \frac{1}{(1+\kappa Y)} \frac{\partial}{\partial X}(\acute{\mu}^*-\frac{2\mu^*}{3})\{\frac{\partial \bar{V}}{\partial y}$$

$$+ \frac{1}{(R \pm Y\sin\alpha)}[\frac{\bar{U}}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm \bar{V}\sin\alpha]\} + \frac{(\acute{\mu}^*+4/3\mu^*)}{(1+\kappa Y)}\{\frac{1}{(1+\kappa Y)}[\frac{\partial^2 \bar{U}}{\partial X^2}$$

$$- \frac{Y}{(1+\kappa Y)} \frac{d\kappa}{dX} \frac{\partial \bar{U}}{\partial X} + \frac{\bar{V}d\kappa}{dX} - \frac{\kappa Y\bar{V}}{(1+\kappa Y)} \frac{d\kappa}{dX}] + \frac{1}{(R \pm Y\sin\alpha)}[\frac{1}{(1+\kappa Y)}(\bar{U}[\frac{d^2R}{dX^2}$$

$$\mp Y\sin\alpha(\frac{d\alpha}{dX})^2 \pm Y\cos\alpha\frac{d^2\alpha}{dX^2}] + [\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}][\frac{\partial \bar{U}}{\partial X} - \frac{\bar{U}Y}{(1+\kappa Y)} \frac{d\kappa}{dX}])$$

$$\pm \bar{V}\cos\alpha\frac{d\alpha}{dX} - (\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})[\frac{\bar{U}}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm \bar{V}\sin\alpha]/(R \pm Y\sin\alpha)]\}$$

$$+ \frac{(\acute{\mu}^*+1/3\mu^*)}{(1+\kappa Y)}[\frac{\partial^2 \bar{V}}{\partial Y \partial X} \pm \frac{\sin\alpha}{(R \pm Y\sin\alpha)} \frac{\partial \bar{V}}{\partial X}] + \frac{(\acute{\mu}^*+7/3\mu^*)}{(1+\kappa Y)^2} \frac{\kappa\partial \bar{V}}{\partial X} + \frac{\partial \bar{\mu}}{\partial Y} \frac{\partial U^*}{\partial Y}$$

$$+ \frac{\partial \mu^*}{\partial Y} \frac{\partial \bar{U}}{\partial Y} + \frac{1}{(1+\kappa Y)} \frac{\partial \bar{\mu}}{\partial Y}(\frac{\partial V^*}{\partial X} - \kappa U^*) + \frac{1}{(1+\kappa Y)} \frac{\partial \mu^*}{\partial Y}(\frac{\partial \bar{V}}{\partial X} - \kappa\bar{U}) + \frac{\bar{\mu}\partial^2 U^*}{\partial Y^2}$$

$$+ \frac{\overline{\mu}\kappa}{(1+\kappa Y)} \frac{\partial U^*}{\partial Y} - \frac{\overline{\mu}\kappa^2 U^*}{(1+\kappa Y)^2} + \frac{\overline{\mu}}{(R \pm Y\sin\alpha)}\left[\frac{R_L^{\,2}}{(R \pm Y\sin\alpha)} \frac{\partial^2 U^*}{\partial Z^2} \pm \frac{\sin\alpha \partial U^*}{\partial Y}\right.$$

$$\left. \pm \frac{\kappa\sin\alpha U^*}{(1+\kappa Y)}\right] + \frac{\mu^* \partial^2 \overline{U}}{\partial Y^2} + \frac{\mu^*\kappa}{(1+\kappa Y)} \frac{\partial \overline{U}}{\partial Y} - \frac{\mu^*\kappa^2 \overline{U}}{(1+\kappa Y)^2} \pm \frac{\mu^*\sin\alpha}{(R \pm Y\sin\alpha)}\left[\frac{\partial \overline{U}}{\partial Y} + \frac{\kappa \overline{U}}{(1+\kappa Y)}\right]$$

$$- \rho^* g\sin\alpha \tag{2.3-1}$$

$$\frac{\overline{\rho}\partial V^*}{\partial t} + \overline{\rho}\left[\frac{1}{(1+\kappa Y)}\left(\frac{\overline{U}\partial V^*}{\partial X} - 2\kappa\overline{U}U^* + \frac{U^*\partial \overline{V}}{\partial X}\right) + \frac{\overline{V}\partial V^*}{\partial Y} + \frac{V^*\partial \overline{V}}{\partial Y}\right] + \rho^*\left[\frac{1}{(1+\kappa Y)}\left(\frac{\overline{U}\partial \overline{V}}{\partial X}\right.\right.$$

$$\left.\left. - \kappa\overline{U}^2\right) + \frac{\overline{V}\partial \overline{V}}{\partial Y}\right] = \frac{-\partial P^*}{\partial Y} + \frac{\partial}{\partial Y}\left(\frac{\overline{\mu}-2\overline{\mu}}{3}\right)\left\{\frac{1}{(1+\kappa Y)}\left(\frac{\partial U^*}{\partial X} + \kappa V^*\right)\right.$$

$$+ \frac{1}{(R \pm Y\sin\alpha)}\left[\frac{U^*}{(1+\kappa Y)}\left(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX}\right) \pm V^*\sin\alpha + R_L\frac{\partial W^*}{\partial Z}\right]\Big\}$$

$$+ \frac{\partial}{\partial Y}\left(\frac{\overline{\mu}+4\overline{\mu}}{3}\right)\frac{\partial V^*}{\partial Y} + \left(\frac{\overline{\mu}+4\overline{\mu}}{3}\right)\left\{\frac{\partial^2 V^*}{\partial Y^2} + \frac{\kappa}{(1+\kappa Y)} \frac{\partial V^*}{\partial Y} - \frac{\kappa^2 V^*}{(1+\kappa Y)^2}\right.$$

$$\pm \frac{1}{(R \pm Y\sin\alpha)}\left[\frac{U^*\cos\alpha}{(1+\kappa Y)} \frac{d\alpha}{dY} + \frac{\sin\alpha\partial V^*}{\partial Y}\right] - \frac{1}{(R \pm Y\sin\alpha)^2}\left[V^*\sin^2\alpha\right.$$

$$\left.\left. \pm \frac{U^*\sin\alpha}{(1+\kappa Y)}\left(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX}\right)\right]\right\} + \left(\frac{\overline{\mu}+1\overline{\mu}}{3}\right)\left\{\frac{1}{(1+\kappa Y)} \frac{\partial^2 U^*}{\partial X \partial Y} + \frac{1}{(R \pm Y\sin\alpha)}\left[\frac{1}{(1+\kappa Y)}\left(\frac{dR}{dX}\right.\right.\right.$$

$$\left.\left. \pm \frac{Y\cos\alpha d\alpha}{dX}\right)\frac{\partial U^*}{\partial Y} + R_L\frac{\partial^2 W^*}{\partial Z \partial Y}\right]\Big\} - \left(\frac{\overline{\mu}+7\overline{\mu}}{3}\right)\left\{\frac{\kappa}{(1+\kappa Y)^2} \frac{\partial U^*}{\partial X}\right.$$

$$+ \frac{1}{(R\pm Y\sin\alpha)}\left[\frac{\kappa U^*}{(1+\kappa Y)^2}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right) \pm \frac{R_L\sin\alpha}{(R\pm Y\sin\alpha)}\frac{\partial W^*}{\partial Z}\right]\}$$

$$+ \frac{\partial}{\partial Y}\left(\acute{\mu}^*-\frac{2\mu^*}{3}\right)\{\frac{1}{(1+\kappa Y)}\left(\frac{\partial\overline{U}}{\partial X} + \kappa\overline{V}\right) + \frac{1}{(R\pm Y\sin\alpha)}\left[\frac{\overline{U}}{(1+\kappa Y)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\right.$$

$$\left.\pm \overline{V}\sin\alpha\right]\} + \frac{\partial}{\partial Y}\left(\acute{\mu}^*+\frac{4\mu^*}{3}\right)\frac{\partial\overline{V}}{\partial Y} + \left(\acute{\mu}^*+\frac{4\mu^*}{3}\right)\{\frac{\partial^2\overline{V}}{\partial Y^2} - \frac{\kappa^2\overline{V}}{(1+\kappa Y)^2} + \frac{\kappa}{(1+\kappa Y)}\frac{\partial\overline{V}}{\partial Y}$$

$$\pm \frac{1}{(R\pm Y\sin\alpha)}\left[\frac{\overline{U}\cos\alpha}{(1+\kappa Y)}\frac{d\alpha}{dX} + \sin\alpha\frac{\partial\overline{V}}{\partial Y}\right] - \frac{1}{(R\pm Y\sin\alpha)^2}\left[\overline{V}\sin^2\alpha \pm \frac{\overline{U}\sin\alpha}{(1+\kappa Y)}\left(\frac{dR}{dX}\right.\right.$$

$$\left.\left.\pm Y\cos\alpha\frac{d\alpha}{dX}\right)\right]\} + \frac{(\acute{\mu}^*+1/3\mu^*)}{(1+\kappa Y)}\left[\frac{\partial^2\overline{U}}{\partial X\partial Y} + \frac{1}{(R\pm Y\sin\alpha)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\frac{\partial\overline{U}}{\partial Y}\right]$$

$$- \frac{(\acute{\mu}^*+7/3\mu^*)}{(1+\kappa Y)^2}\kappa\left[\frac{\partial\overline{U}}{\partial X} + \frac{\overline{U}}{(R\pm Y\sin\alpha)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\right] + \frac{1}{(1+\kappa Y)^2}\frac{\partial\overline{\mu}}{\partial X}\left(\frac{\partial V^*}{\partial X}\right.$$

$$\left.- \kappa U^*\right) + \frac{1}{(1+\kappa Y)^2}\frac{\partial\mu^*}{\partial X}\left(\frac{\partial\overline{V}}{\partial X} - \kappa\overline{U}\right) + \frac{1}{(1+\kappa Y)}\left(\frac{\partial\overline{\mu}}{\partial X}\frac{\partial U^*}{\partial Y} + \frac{\partial\mu^*}{\partial X}\frac{\partial\overline{U}}{\partial Y}\right)$$

$$+ \frac{\overline{\mu}}{(1+\kappa Y)^2}\left[\frac{\partial^2 V^*}{\partial X^2} + \frac{Y}{(1+\kappa Y)}\frac{d\kappa}{dX}\left(\kappa U^* - \frac{\partial V^*}{\partial X}\right)\right] + \frac{\overline{\mu}R_L{}^2}{(R\pm Y\sin\alpha)^2}\frac{\partial^2 V^*}{\partial Z^2}$$

$$+ \frac{\overline{\mu}}{(1+\kappa Y)^2(R\pm Y\sin\alpha)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\frac{\partial V^*}{\partial X} + \frac{\mu^*}{(1+\kappa Y)^2}\left[\frac{\partial^2\overline{V}}{\partial X^2}\right.$$

$$\left.+ \frac{Y}{(1+\kappa Y)}\frac{d\kappa}{dX}\left(\kappa\overline{U} - \frac{\partial\overline{V}}{\partial X}\right)\right] + \frac{\mu^*}{(1+\kappa Y)^2(R\pm Y\sin\alpha)}\left(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}\right)\frac{\partial\overline{V}}{\partial X} - \rho^* g\cos\alpha$$

$$\bar{\rho}\frac{\partial W^*}{\partial t} + \bar{\rho}\{\frac{\bar{U}}{(1+\kappa Y)}\frac{\partial W^*}{\partial X} + \bar{V}\frac{\partial W^*}{\partial Y} + \frac{W^*}{(R\pm Y\sin\alpha)}[\frac{\bar{U}}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm \bar{V}\sin\alpha]\}$$

$$= \frac{-\partial P^*}{\partial Z} + \frac{(\bar{\acute{\mu}}+4/3\bar{\mu})R_L}{(R\pm Y\sin\alpha)}\{\frac{\kappa}{(1+\kappa Y)}\frac{\partial V^*}{\partial Z} + \frac{1}{(R\pm Y\sin\alpha)}[\frac{1}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\frac{\partial U^*}{\partial Z}$$

$$\pm \sin\alpha\frac{\partial V^*}{\partial Z} + R_L\frac{\partial^2 W^*}{\partial Z^2}]\} + \frac{(\bar{\acute{\mu}}+1/3\bar{\mu})R_L}{(R\pm Y\sin\alpha)}[\frac{1}{(1+\kappa Y)}\frac{\partial^2 U^*}{\partial X\partial Z} + \frac{\partial^2 V^*}{\partial Y\partial Z}]$$

$$+ \frac{R_L}{(R\pm Y\sin\alpha)}\frac{\partial}{\partial Z}(\acute{\mu}^* - \frac{2\mu^*}{3})[\frac{1}{(1+\kappa Y)}\frac{\partial\bar{U}}{\partial X} + \frac{\kappa\bar{V}}{(1+\kappa Y)} + \frac{\partial\bar{V}}{\partial Y}] + \frac{\partial\bar{\mu}}{\partial Y}\frac{\partial W^*}{\partial Y}$$

$$+ \frac{R_L}{(R\pm Y\sin\alpha)^2}\frac{\partial}{\partial Z}(\acute{\mu}^* + \frac{4\mu^*}{3})[\frac{\bar{U}}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX}) \pm \bar{V}\sin\alpha]$$

$$+ \frac{1}{(1+\kappa Y)^2}\frac{\partial\bar{\mu}}{\partial X}\frac{\partial W^*}{\partial X} + \frac{R_L}{(R\pm Y\sin\alpha)}\{\frac{1}{(1+\kappa Y)}\frac{\partial\bar{\mu}}{\partial X}[\frac{\partial U^*}{\partial Z} - \frac{W^*}{R_L(1+\kappa Y)}(\frac{dR}{dX}$$

$$\pm Y\cos\alpha\frac{d\alpha}{dX})] + \frac{\partial\bar{\mu}}{\partial Y}(\frac{\partial V^*}{\partial Z} \mp \frac{W^*\sin\alpha}{R_L})\} + \frac{\bar{\mu}}{(1+\kappa Y)}\{\frac{1}{(R\pm Y\sin\alpha)}[\frac{W^*}{(1+\kappa Y)}(\frac{d^2R}{dX^2}$$

$$\mp Y\sin\alpha(\frac{d\alpha}{dX})^2 \pm Y\cos\alpha\frac{d^2\alpha}{dX^2}) + \frac{1}{(1+\kappa Y)}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})(\frac{\partial W^*}{\partial X} - \frac{YW^*}{(1+\kappa Y)}\frac{d\kappa}{dX})$$

$$\pm (1+\kappa Y)\sin\alpha\frac{\partial W^*}{\partial Y} \pm \kappa\sin\alpha\, W^* - R_L\kappa\frac{\partial V^*}{\partial Z}] - \frac{1}{(R\pm Y\sin\alpha)^2}[\frac{W^*}{(1+\kappa Y)}(\frac{dR}{dX}$$

$$\pm Y\cos\alpha\frac{d\alpha}{dX})^2 - R_L(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\frac{\partial U^*}{\partial Z} + (1+\kappa Y)W^*\sin^2\alpha$$

$$\mp R_L(1+\kappa Y)\sin\alpha\frac{\partial V^*}{\partial Z}]\}$$

$$\overline{\rho C}_p\frac{\partial\Theta^*}{\partial t} + \overline{\rho C}_p[\frac{\overline{U}}{(1+\kappa Y)}\frac{\partial\Theta^*}{\partial X} + \frac{\overline{V}\partial\Theta^*}{\partial Y} + \frac{U^*}{(1+\kappa Y)}\frac{\partial\overline{\Theta}}{\partial X} + \frac{V^*\partial\overline{\Theta}}{\partial Y}] + (\rho^*\overline{C}_p + \overline{\rho C}_p^*)[\frac{\overline{U}}{(1+\kappa Y)}\frac{\partial\overline{\Theta}}{\partial X}$$

$$+ \frac{\overline{V}\partial\overline{\Theta}}{\partial Y}] + \frac{T}{\rho}(\frac{\partial\rho}{\partial T})_p[\frac{\partial P^*}{\partial t} + \frac{\overline{U}}{(1+\kappa Y)}\frac{\partial P^*}{\partial X} + \frac{U^*}{(1+\kappa Y)}\frac{\partial\overline{P}}{\partial X} + \frac{V^*\partial\overline{P}}{\partial Y} + \frac{\overline{V}\partial P^*}{\partial Y}]$$

$$= \overline{k}\{\frac{\partial^2\Theta^*}{\partial Y^2} + \frac{\kappa}{(1+\kappa Y)}\frac{\partial\Theta^*}{\partial Y} + \frac{1}{(1+\kappa Y)^2}\frac{\partial^2\Theta^*}{\partial X^2} - \frac{Y}{(1+\kappa Y)^3}\frac{d\kappa}{dX}\frac{\partial\Theta^*}{\partial X}$$

$$+ \frac{1}{(R\pm Y\sin\alpha)}[\frac{1}{(1+\kappa Y)^2}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\frac{\partial\Theta^*}{\partial X} \pm \sin\alpha\frac{\partial\Theta^*}{\partial Y} + \frac{R_L{}^2}{(R\pm Y\sin\alpha)}\frac{\partial^2\Theta^*}{\partial Z^2}]\}$$

$$+ k^*\{\frac{\partial^2\overline{\Theta}}{\partial Y^2} + \frac{\kappa}{(1+\kappa Y)}\frac{\partial\overline{\Theta}}{\partial Y} + \frac{1}{(1+\kappa Y)^2}\frac{\partial^2\overline{\Theta}}{\partial X^2} - \frac{Y}{(1+\kappa Y)^3}\frac{d\kappa}{dX}\frac{\partial\overline{\Theta}}{\partial X}$$

$$+ \frac{1}{(R\pm Y\sin\alpha)}[\frac{1}{(1+\kappa Y)^2}(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\frac{\partial\overline{\Theta}}{\partial X} \pm \sin\alpha\frac{\partial\overline{\Theta}}{\partial Y}]\} + \frac{\partial\overline{k}}{\partial Y}\frac{\partial\Theta^*}{\partial Y}$$

$$+ \frac{1}{(1+\kappa Y)^2}\frac{\partial\overline{k}}{\partial X}\frac{\partial\Theta^*}{\partial X} + \frac{\partial k^*}{\partial Y}\frac{\partial\overline{\Theta}}{\partial Y} + \frac{1}{(1+\kappa Y)^2}\frac{\partial k^*}{\partial X}\frac{\partial\overline{\Theta}}{\partial X} + \frac{2(\overline{\mu}-2/3\overline{\mu})}{(1+\kappa Y)}\{\frac{\partial\overline{V}}{\partial Y}\frac{\partial U^*}{\partial X}$$

$$+ \kappa V^*\frac{\partial\overline{V}}{\partial Y} + \frac{\partial\overline{U}}{\partial X}\frac{\partial V^*}{\partial Y} + \kappa\overline{V}\frac{\partial V^*}{\partial Y} + \frac{1}{(R\pm Y\sin\alpha)}[(\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})\frac{\overline{U}\partial V^*}{\partial Y}$$

$$\pm \overline{V}\sin\alpha\frac{\partial U^*}{\partial X} \pm 2\kappa\overline{V}V^*\sin\alpha + (\frac{dR}{dX} \pm Y\cos\alpha\frac{d\alpha}{dX})U^*\frac{\partial\overline{V}}{\partial Y} + R_L\frac{\partial W^*}{\partial Z}\frac{\partial\overline{U}}{\partial X} + R_L\kappa\overline{V}\frac{\partial W^*}{\partial Z}$$

$$\pm \frac{V^*\sin\alpha\partial\overline{U}}{\partial X} + \frac{1}{(1+\kappa Y)}(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX})(\frac{\overline{U}\partial U^*}{\partial X} + \kappa\overline{U}V^* + \frac{U^*\partial\overline{U}}{\partial X} + \kappa U^*\overline{V})]\}$$

$$+ \frac{2(\overline{\acute{\mu}}-2/3\overline{\mu})}{(R\pm Y\sin\alpha)}(R_L\frac{\partial W^*}{\partial Z}\frac{\partial\overline{V}}{\partial Y} \pm \frac{\overline{V}\sin\alpha\partial V^*}{\partial Y} \pm \frac{V^*\sin\alpha\partial\overline{V}}{\partial Y}) + 2(\overline{\acute{\mu}}+4/3\overline{\mu})\{\frac{\partial\overline{V}}{\partial Y}\frac{\partial V^*}{\partial Y}$$

$$+ \frac{1}{(1+\kappa Y)^2}(\frac{\partial\overline{U}}{\partial X}\frac{\partial U^*}{\partial X} + \frac{\kappa\overline{V}\partial U^*}{\partial X} + \frac{\kappa V^*\partial\overline{U}}{\partial X} + \kappa^2\overline{V}V^*) + \frac{1}{(R\pm Y\sin\alpha)^2}[\overline{V}V^*\sin^2\alpha$$

$$\pm \frac{R_L\overline{V}\sin\alpha\partial W^*}{\partial Z} + \frac{1}{(1+\kappa Y)}(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX})(\frac{R_L\overline{U}\partial W^*}{\partial Z} \pm U^*\overline{V}\sin\alpha \pm \overline{U}V^*\sin\alpha)$$

$$+ \frac{U^*\overline{U}}{(1+\kappa Y)^2}(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX})^2]\} + \frac{2(\acute{\mu}^*-2/3\mu^*)}{(1+\kappa Y)}\{\frac{\partial\overline{V}}{\partial Y}\frac{\partial\overline{U}}{\partial X} + \frac{\kappa\overline{V}\partial\overline{V}}{\partial Y}$$

$$+ \frac{1}{(R\pm Y\sin\alpha)}[(\frac{dR}{dX} \pm \frac{\cos\alpha d\alpha}{dX})\frac{\overline{U}\partial\overline{V}}{\partial Y} \pm \kappa\overline{V}^2\sin\alpha \pm \frac{\overline{V}\sin\alpha\partial\overline{U}}{\partial X} + \frac{1}{(1+\kappa Y)}(\frac{dR}{dX}$$

$$\pm \frac{Y\cos\alpha d\alpha}{dX})(\frac{\overline{U}\partial\overline{U}}{\partial X} + \kappa\overline{U}\overline{V}]\} \pm \frac{2(\acute{\mu}^*-2/3\mu^*)}{(R\pm Y\sin\alpha)}\frac{\overline{V}\sin\alpha\partial\overline{V}}{\partial Y} + (\acute{\mu}^*+\frac{4\mu^*}{3})\{(\frac{\partial\overline{V}}{\partial Y})^2$$

$$+ \frac{1}{(1+\kappa Y)^2}[(\frac{\partial\overline{U}}{\partial X})^2 + \frac{2\kappa\overline{V}\partial\overline{U}}{\partial X} + \kappa^2\overline{V}^2] + \frac{1}{(R\pm Y\sin\alpha)^2}[\overline{V}^2\sin^2\alpha \pm \frac{2\overline{U}\overline{V}\sin\alpha}{(1+\kappa Y)}(\frac{dR}{dX}$$

$$\pm \frac{Y\cos\alpha d\alpha}{dX}) + \frac{\overline{U}^2}{(1+\kappa Y)^2}(\frac{dR}{dX} \pm \frac{Y\cos\alpha d\alpha}{dX})^2]\} + 2\overline{\mu}[\frac{\partial\overline{U}}{\partial Y}\frac{\partial U^*}{\partial Y} + \frac{1}{(1+\kappa Y)}(\frac{\partial\overline{U}}{\partial Y}\frac{\partial V^*}{\partial X}$$

$$+ \frac{\partial\overline{V}}{\partial X}\frac{\partial U^*}{\partial Y} - \frac{\kappa\overline{U}\partial U^*}{\partial Y} - \frac{\kappa U^*\partial\overline{U}}{\partial Y}) + \frac{1}{(1+\kappa Y)^2}(\frac{\partial\overline{V}}{\partial X}\frac{\partial V^*}{\partial X} - \frac{\kappa\overline{U}\partial V^*}{\partial X} - \frac{\kappa U^*\partial\overline{V}}{\partial X}$$

$$+ \kappa^2\overline{U}U^*)] + \mu^*\{(\frac{\partial\overline{U}}{\partial Y})^2 + \frac{2}{(1+\kappa Y)}(\frac{\partial\overline{U}}{\partial Y}\frac{\partial\overline{V}}{\partial X} - \frac{\kappa\overline{U}\partial\overline{U}}{\partial Y}) + \frac{1}{(1+\kappa Y)^2}[(\frac{\partial\overline{V}}{\partial X})^2$$

$$- \frac{2\kappa\overline{U}\partial\overline{V}}{\partial X} + \kappa^2\overline{U}^2]\}$$

The semi-infinite extent of the region in the X direction and the lack of distinct boundaries in the Z direction present difficulties in attempting to establish a set of boundary conditions for the above set of equations. However, even supposing that the boundary conditions are available, the complexity of the above equations makes any solution attempt very difficult. Therefore, it is desirable to seek some simplification to the above equations.

2.3-2 Simplified Disturbance Equations

In appendix D, the set of equations 2.3-1 is normalized and an order-of-magnitude analysis is applied. The end result is the set of equations D-1 which constitutes one form of the simplified disturbance equations. The mean-flow quantities appearing in the equations are assumed to be known. However, the equations cannot be solved without having auxiliary relationships for M*, c* and K* in terms of the other disturbance variables.

Assuming that these auxiliary relationships are available, then a suitable representation must be found for the time dependence, for the semi-infinite extent of the region in the X direction and for the lack of boundaries in the Z direction. This problem is approached by assuming that the disturbance variables are proportional to $\exp\,[i(\lambda x^* + \Omega z - B\tau)]$. Using this assumed proportionality, the disturbance variables are bounded

as X or Z goes to infinity, and the stability of the flow can be analysed by considering a single component of a Fourier series in x* and z. The real parts of λ and Ω, λ_r and Ω_r, are the wave numbers in the x* and z directions, respectively, and the real part of B, B_r, is the frequency of the disturbance. The imaginary parts of λ, Ω and B, λ_i, Ω_i and B_i, are amplification factors. The disturbance grows with increasing x*, z and τ if λ_i and Ω_i are negative and B_i is positive. If λ_i, Ω_i and B_i are all zero, the disturbance neither amplifies or attenuates and the flow is said to be neutrally stable.

For the stability analyses of free-convection flows, Knowles and Gebhart [36] have shown that for $\alpha = \pi/2$ free-convection flows subjected to two-dimensional disturbances have a lower critical Rayleigh number than the same flows subjected to three-dimensional disturbances. However, since the present analysis is for all α, an attempt is made to solve the disturbance equations as they apply for two types of disturbances. The first type is a two-dimensional disturbance in the form of a wave which travels in the X direction with an amplitude dependent only on y. The second type is a three-dimensional disturbance in the form of a set of roll vortices having their axes parallel to the X direction. These vortices are periodic in z and have amplitudes which depend only on y. In addition, these vortices are assumed to be independent of x*, but this does not imply that u* is zero.

2.3-3 Tollmien-Schlichting Wave Disturbances

In forced-convection stability analyses, two-dimensional wave disturbances similar to those described above are called Tollmien-Schlichting waves. This terminology is also applied in the present stability analysis.

For Tollmien-Schlichting wave disturbances, the set of equations D-1, with the chosen function for $H(\omega)$, can be simplified to the following:

$$\frac{\partial\gamma_d^*}{\partial\tau} + (1+\bar{\gamma}_d)[\frac{\partial v^*}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\partial u^*}{\partial x^*} + Aqv^*) \pm \frac{v^*J\sin\alpha}{(j\pm Jy\sin\alpha)}] + \frac{v^*\partial\gamma_d}{\partial y} + \frac{\bar{u}}{(1+Aqy)}\frac{\partial\gamma_d^*}{\partial x^*}$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{1}{(j\pm Jy\sin\alpha)}(\frac{dj}{d\bar{x}} \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}})[\frac{(1+\bar{\gamma}_d)u^*}{(1+Aqy)} + \frac{\bar{\gamma}_d^*\bar{u}}{(1+Aqy)}]$$

$$\pm \frac{\gamma_d^*\bar{v}J\sin\alpha}{(j\pm Jy\sin\alpha)} + \frac{1}{(1+Aqy)}[\gamma_d^*(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}) + \frac{u^*\partial\bar{\gamma}_d}{\partial\bar{x}}] + \gamma_d^*\frac{\partial\bar{v}}{\partial y} + \frac{\bar{v}\partial\gamma_d^*}{\partial y}\} = 0$$

$$(1+\bar{\gamma}_d)[\frac{1}{(1+Aqy)}(\frac{\partial^2 v^*}{\partial x^*\partial\tau} - \frac{Aq\partial u^*}{\partial\tau} - \frac{2Aqv^*\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}\partial v^*}{\partial y} - \frac{\partial\bar{u}}{\partial y}\frac{\partial u^*}{\partial x^*} - \frac{\bar{u}\partial^2 u^*}{\partial y\partial x^*}) - \frac{\partial^2 u^*}{\partial y\partial\tau}$$

$$- \frac{v^*\partial^2\bar{u}}{\partial y^2} - \frac{\partial v^*}{\partial y}\frac{\partial\bar{u}}{\partial y} - \frac{\bar{u}}{(1+Aqy)^2}(\frac{2Aq\partial u^*}{\partial x^*} - \frac{\partial^2 v^*}{\partial x^{*2}})] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\partial u^*}{\partial\tau} + \frac{v^*\partial\bar{u}}{\partial y}$$

$$+ \frac{\bar{u}}{(1+Aqy)}(\frac{\partial u^*}{\partial x^*} + Aqv^*)] - \frac{Aq\bar{u}^2}{(1+Aqy)^2}\frac{\partial\gamma_d^*}{\partial x^*} + Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{1}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}[\frac{\partial v^*}{\partial\tau}$$

$$+ \frac{\bar{u}}{(1+Aqy)}(\frac{\partial v^*}{\partial x^*} - 2Aqu^*)] - (1+\bar{\gamma}_d)[\frac{\bar{v}\partial^2 u^*}{\partial y^2} + \frac{\partial\bar{v}}{\partial y}\frac{\partial u^*}{\partial y} + \frac{1}{(1+Aqy)}(\frac{2Aq\bar{v}\partial u^*}{\partial y}$$

$$- \frac{\bar{v}\partial^2 v^*}{\partial x^*\partial y} - \frac{\partial v^*}{\partial x^*}\frac{\partial\bar{v}}{\partial y} + \frac{u^*\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial u^*}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aqu^*\partial\bar{v}}{\partial y}) + \frac{1}{(1+Aqy)^2}(\frac{2A\bar{u}u^*dq}{dx}$$

$$+ \frac{2Aqu^*\partial\bar{u}}{\partial\bar{x}} - \frac{\partial v^*}{\partial x^*}\frac{\partial\bar{u}}{\partial\bar{x}}) + \frac{Ay\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}}(\frac{\partial v^*}{\partial x^*} - 2Aqu^*)] - \gamma_d{}^*[\frac{\bar{v}\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aq\bar{u}\partial\bar{v}}{\partial y} + \frac{2Aq\bar{v}\partial\bar{u}}{\partial y}) + \frac{A\bar{u}}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}}$$

$$+ \frac{2q\partial\bar{u}}{\partial\bar{x}}) - \frac{A^2qy\bar{u}^2}{(1+Aqy)^3}\frac{dq}{d\bar{x}}] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\bar{v}\partial u^*}{\partial y} + \frac{u^*}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]$$

$$- \frac{\partial\gamma_d{}^*}{\partial y}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{\partial^2\bar{M}}{\partial y^2}[\frac{1}{(1+Aqy)}(Aqu^* - \frac{\partial v^*}{\partial x^*}) - \frac{\partial u^*}{\partial y}] - \frac{\partial\bar{M}}{\partial y}[\frac{2\partial^2u^*}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial u^*}{\partial y} + \frac{1}{(1+Aqy)^2}(\frac{2\partial^2u^*}{\partial x^{*2}} + \frac{3Aq\partial v^*}{\partial x^*} - A^2q^2u^*)]$$

$$\mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\bar{M}}{\partial y}[\frac{1}{(1+Aqy)}(\frac{\partial v^*}{\partial x^*} + Aqu^*) + \frac{\partial u^*}{\partial y}] + \frac{1}{(1+Aqy)^2}\frac{\partial^2M^*}{\partial x^{*2}}[\frac{\partial\bar{u}}{\partial y}$$

$$- \frac{Aq\bar{u}}{(1+Aqy)}] + \frac{\partial^2M^*}{\partial y^2}[\frac{Aq\bar{u}}{(1+Aqy)} - \frac{\partial\bar{u}}{\partial y}] - \frac{\partial M^*}{\partial y}[\frac{2\partial^2\bar{u}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q\bar{u}}{(1+Aqy)^2}]$$

$$\mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial M^*}{\partial y}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}] - \bar{M}[\frac{\partial^3u^*}{\partial y^3} + \frac{1}{(1+Aqy)}(\frac{2Aq\partial^2u^*}{\partial y^2} - \frac{\partial^3v^*}{\partial y^2\partial x^*})$$

$$+ \frac{1}{(1+Aqy)^2}(\frac{\partial^3u^*}{\partial x^{*2}\partial y} + \frac{Aq\partial^2v^*}{\partial y\partial x^*} - \frac{A^2q^2\partial u^*}{\partial y}) - \frac{1}{(1+Aqy)^3}(\frac{\partial^3v^*}{\partial x^{*3}} - \frac{Aq\partial^2u^*}{\partial x^{*2}}$$

$$+ \frac{A^2q^2\partial v^*}{\partial x^*} - A^3q^3u^*)] \mp \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 u^*}{\partial y^2} + \frac{1}{(1+Aqy)}(\frac{2Aq\partial u^*}{\partial y} - \frac{\partial^2 v^*}{\partial y\partial x^*})]$$

$$+ \frac{\bar{M}}{(j\pm Jy\sin\alpha)^2}[\frac{J^2\sin^2\alpha\partial u^*}{\partial y} - \frac{J^2\sin^2\alpha}{(1+Aqy)}(\frac{\partial v^*}{\partial x^*} - Aqu^*)] - M^*[\frac{\partial^3\bar{u}}{\partial y^3}$$

$$+ \frac{2Aq}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y^2} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial y} + \frac{A^3q^3\bar{u}}{(1+Aqy)^3}] \mp \frac{M^*J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{2Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y}] + \frac{M^*J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\} + \frac{\sigma Ra_L^{-\omega^2}}{\beta\bar{\Theta}_c}[\frac{\partial\gamma_d^*\sin\alpha}{\partial y}$$

$$+ \frac{\gamma_d^*Aq\sin\alpha}{(1+Aqy)} - \frac{\cos\alpha}{(1+Aqy)}\frac{\partial\gamma_d^*}{\partial x^*} + Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\gamma_d^*\sin\alpha}{(1+Aqy)}\frac{d\alpha}{d\bar{x}}]$$

(2.3-2)

$$(1+\bar{\gamma}_d)\bar{c}(1+a)[\frac{\partial\theta^*}{\partial\tau} + \frac{\bar{u}}{(1+Aqy)}\frac{\partial\theta^*}{\partial x^*} + \frac{v^*\partial\bar{\theta}}{\partial y}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}\{(1+\bar{\gamma}_d)\bar{c}[\frac{1}{(1+Aqy)}\frac{da}{d\bar{x}}(\bar{u}\theta^*$$

$$+ u^*\bar{\theta}) + \frac{\bar{v}(1+a)\partial\theta^*}{\partial y} + \frac{u^*(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{x}}] + \gamma_d^*\bar{c}[\frac{\bar{u}(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}}$$

$$+ \frac{\bar{v}(1+a)\partial\bar{\theta}}{\partial y}] + (1+\bar{\gamma}_d)c^*[\frac{\bar{u}(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}} + \frac{\bar{v}(1+a)\partial\bar{\theta}}{\partial y}]\}$$

$$= (1+a)\,Ra_L^{-\frac{1}{5}(1+\omega^2)}\{K^*[\frac{\partial^2\bar{\theta}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\bar{\theta}}{\partial y}] + \bar{K}[\frac{\partial^2\theta^*}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial\theta^*}{\partial y} + \frac{1}{(1+Aqy)^2}\frac{\partial^2\theta^*}{\partial x^{*2}} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\theta^*}{\partial y}] + \frac{\partial K^*\partial\bar{\theta}}{\partial y\,\partial y} + \frac{\partial\bar{K}}{\partial y}\frac{\partial\theta^*}{\partial y}\}$$

Assuming that the disturbance variables are proportional to $\exp[i(\lambda x^*-B\tau)]$ and that the stability of the flow can be analysed by considering a single component of a Fourier series in x^*, the disturbance variables are expressed as follows:

$$u^*(x^*, y, \tau) = \delta_1(y)\, e^{i(\lambda x^* - B\tau)}$$

$$v^*(x^*, y, \tau) = \delta_2(y)\, e^{i(\lambda x^* - B\tau)}$$

$$\theta^*(x^*, y, \tau) = s(y)\, e^{i(\lambda x^* - B\tau)}$$

$$\gamma_d^*(x^*, y, \tau) = r(y)\, e^{i(\lambda x^* - B\tau)}$$

$$M^*(x^*, y, \tau) = \Lambda(y)\, e^{i(\lambda x^* - B\tau)}$$

$$c^*(x^*, y, \tau) = \Delta(y)\, e^{i(\lambda x^* - B\tau)}$$

$$K^*(x^*, y, \tau) = \Upsilon(y)\, e^{i(\lambda x^* - B\tau)}$$

Substituting these expressions into the set of equations 2.3-2 and eliminating the exponential factor from each term, the following set of equations is obtained:

$$\left\{\frac{i\lambda(1+\bar{\gamma}_d)}{(1+Aqy)} + \frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\left[\frac{\partial\bar{\gamma}_d}{\partial\bar{x}} + \frac{(1+\bar{\gamma}_d)}{(j\pm Jysin\alpha)}\left(\frac{dj}{d\bar{x}} \pm Jycos\alpha\frac{d\alpha}{d\bar{x}}\right)\right]\right\}\delta_1 + \left\{(1+\bar{\gamma}_d)\left[\frac{Aq}{(1+Aqy)}\right.\right.$$

$$\left.\left.\pm \frac{Jsin\alpha}{(j\pm Jysin\alpha)}\right] + \frac{\partial\bar{\gamma}_d}{\partial y}\right\}\delta_2 + (1+\bar{\gamma}_d)\frac{d\delta_2}{dy} + \left\{\frac{i\lambda\bar{u}}{(1+Aqy)} - iB\right.$$

$$+ \frac{Ra_L^{-\frac{1}{5}}(1+\omega^2)}{(j\pm Jysin\alpha)} \left[\frac{\bar{u}}{(1+Aqy)}\left(\frac{dj}{d\bar{x}} \pm Jycos\alpha\frac{d\alpha}{d\bar{x}}\right) \pm \bar{v}Jsin\alpha\right]$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)} \left[\frac{1}{(1+Aqy)}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}\right) + \frac{\partial\bar{v}}{\partial y}\right]\}r + Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{dr}{dy} = 0$$

$$\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{M}\frac{d^3\delta_1}{dy^3} + Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{2\sigma\partial\bar{M}}{\partial y} + \frac{2\sigma\bar{M}Aq}{(1+Aqy)} - (1+\bar{\gamma}_d)\bar{v} \pm \frac{\sigma\bar{M}Jsin\alpha}{(j\pm Jysin\alpha)}\right]\frac{d^2\delta_1}{dy^2}$$

$$+ \{i(1+\bar{\gamma}_d)\left[B - \frac{\lambda\bar{u}}{(1+Aqy)}\right] + Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{\sigma\partial^2\bar{M}}{\partial y^2} + \frac{2\sigma\partial\bar{M}}{\partial y} + \frac{\sigma Aq}{(1+Aqy)}\frac{\partial\bar{M}}{\partial y}\right.$$

$$\pm \frac{\sigma Jsin\alpha}{(j\pm Jysin\alpha)}\frac{\partial\bar{M}}{\partial y} - \frac{\sigma\bar{M}\lambda^2}{(1+Aqy)^2} - \frac{\sigma\bar{M}A^2q^2}{(1+Aqy)^2} \pm \frac{2\sigma\bar{M}Jsin\alpha}{(j\pm Jysin\alpha)}\frac{Aq}{(1+Aqy)}$$

$$\left. - \frac{\sigma\bar{M}J^2sin^2\alpha}{(j\pm Jysin\alpha)^2} - (1+\bar{\gamma}_d)\frac{\partial\bar{v}}{\partial y} - \frac{(1+\bar{\gamma}_d)}{(1+Aqy)}\left(2Aq\bar{v} + \frac{\partial\bar{u}}{\partial\bar{x}}\right) - \frac{\bar{v}\partial\bar{\gamma}_d}{\partial y}\right]\}\frac{d\delta_1}{dy}$$

$$+ \{\frac{(1+\bar{\gamma}_d)}{(1+Aqy)}\left[iAqB - \frac{i\lambda\partial\bar{u}}{\partial y} - \frac{2i\lambda Aq\bar{u}}{(1+Aqy)}\right] + \frac{i\partial\bar{\gamma}_d}{\partial y}\left[B - \frac{\lambda\bar{u}}{(1+Aqy)}\right]$$

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{\sigma Aq}{(1+Aqy)}\frac{\partial^2\bar{M}}{\partial y^2} + \frac{\sigma}{(1+Aqy)^2}(2\lambda^2 + A^2q^2)\frac{\partial\bar{M}}{\partial y}\right.$$

$$\mp \frac{Jsin\alpha}{(j\pm Jysin\alpha)}\frac{\sigma Aq}{(1+Aqy)}\frac{\partial\bar{M}}{\partial y} + \frac{\sigma\bar{M}Aq}{(1+Aqy)}(\lambda^2 - A^2q^2) + \frac{\sigma\bar{M}Aq}{(j\pm Jysin\alpha)^2}\frac{J^2sin^2\alpha}{(1+Aqy)}$$

$$+ \frac{2Aq\bar{u}}{(1+Aqy)^2}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}} + \frac{(1+\bar{\gamma}_d)}{(1+Aqy)}\left(\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{Aq\partial\bar{v}}{\partial y}\right) + \frac{2(1+\bar{\gamma}_d)A}{(1+Aqy)^2}\left(\frac{\bar{u}dq}{d\bar{x}} + \frac{q\partial\bar{u}}{\partial\bar{x}}\right)$$

$$- \frac{2(1+\bar{\gamma}_d)A^2 qy\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}} + \frac{1}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial y}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}\ \delta_1$$

$$- i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\bar{M}\lambda}{(1+Aqy)}\frac{d^2\delta_2}{dy^2} + \{Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{i\sigma\bar{M}\lambda Aq}{(1+Aqy)^2} \mp \frac{i\sigma\bar{M}\lambda J\sin\alpha}{(j\pm Jy\sin\alpha)(1+Aqy)}$$

$$+ \frac{i(1+\bar{\gamma}_d)\lambda\bar{v}}{(1+Aqy)}] - \frac{(1+\bar{\gamma}_d)Aq\bar{u}}{(1+Aqy)} - (1+\bar{\gamma}_d)\frac{\partial\bar{u}}{\partial y}\}\frac{d\delta_2}{dy} + \{(1+\bar{\gamma}_d)[\frac{1}{(1+Aqy)}(\lambda B$$

$$- 2Aq\frac{\partial\bar{u}}{\partial y}) - \frac{\partial^2\bar{u}}{\partial y^2} - \frac{\lambda^2\bar{u}}{(1+Aqy)^2}] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{i\lambda\bar{u}}{(1+Aqy)^2}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}} - \frac{iB}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}} + \frac{i(1+\bar{\gamma}_d)\lambda}{(1+Aqy)}\frac{\partial\bar{v}}{\partial y} + \frac{i(1+\bar{\gamma}_d)\lambda}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial\bar{x}}$$

$$- \frac{i(1+\bar{\gamma}_d)Ay\lambda\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}} + \frac{i\sigma\lambda}{(1+Aqy)}\frac{\partial^2\bar{M}}{\partial y^2} + \frac{3i\sigma Aq\lambda}{(1+Aqy)^2}\frac{\partial\bar{M}}{\partial y} \pm \frac{i\sigma\lambda J\sin\alpha}{(j\pm Jy\sin\alpha)(1+Aqy)}\frac{\partial\bar{M}}{\partial y}$$

$$+ \frac{i\sigma\bar{M}\lambda}{(1+Aqy)^3}(\lambda^2 - A^2q^2) + \frac{i\sigma\bar{M}\lambda J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2(1+Aqy)}]\}\delta_2 - \{\frac{\sigma Ra_L}{\beta\bar{\Theta}_c}^{-\omega^2}\sin\alpha$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}\frac{dr}{dy} + \{\frac{\sigma}{(1+Aqy)}[\frac{Ra_L}{\beta\bar{\Theta}_c}^{-\omega^2}(i\lambda\cos\alpha$$

$$- Aq\sin\alpha) + \frac{Ra_L}{\beta\bar{\Theta}_c}^{-\frac{1}{5}(1+6\omega^2)}\sin\alpha\frac{d\alpha}{d\bar{x}}] - \frac{iAq\lambda\bar{u}^2}{(1+Aqy)^2} - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{v}\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{u}\frac{\partial\bar{v}}{\partial y} + 2Aq\bar{v}\frac{\partial\bar{u}}{\partial y}) + \frac{A\bar{u}}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}}$$

$$+ \frac{2q\partial\bar{u}}{\partial\bar{\bar{x}}}) - \frac{A^2qy\bar{u}^2}{(1+Aqy)^3}\frac{dq}{d\bar{\bar{x}}}]\} \; r + \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}]\frac{d^2\Lambda}{dy^2}$$

$$+ \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{d\Lambda}{dy}[\frac{2\partial^2\bar{u}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q\bar{u}}{(1+Aqy)^2}]$$

$$\pm \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{d\Lambda}{dy}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}] + \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\Lambda[\frac{\partial^3\bar{u}}{\partial y^3}$$

$$+ \frac{2Aq}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y^2} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial y} + \frac{A^3q^3\bar{u}}{(1+Aqy)^3}] \pm \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\Lambda J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{2Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y}] - \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\Lambda J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]$$

$$+ \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\lambda^2\Lambda}{(1+Aqy)}[\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}] = 0 \tag{2.3-3}$$

$$(1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\bar{K}d^2s}{dy^2} + (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{K}Aq}{(1+Aqy)} \pm \frac{\bar{K}J\sin\alpha}{(j\pm Jy\sin\alpha)} + \frac{\partial\bar{K}}{\partial y}$$

$$- (1+\bar{\gamma}_d)\bar{c}\bar{v}]\frac{ds}{dy} + \{i(1+\bar{\gamma}_d)\bar{c}(1+a)[B - \frac{\lambda\bar{u}}{(1+Aqy)}] - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{(1+\bar{\gamma}_d)\bar{c}\bar{u}}{(1+Aqy)}\frac{da}{d\bar{\bar{x}}}$$

$$+ \frac{(1+a)\bar{K}\lambda^2}{(1+Aqy)^2}]\} \; s - Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{(1+\bar{\gamma}_d)\bar{c}}{(1+Aqy)}[\bar{\theta}\frac{da}{d\bar{\bar{x}}} + (1+a)\frac{\partial\bar{\theta}}{\partial\bar{\bar{x}}}] \; \delta_1$$

$$- (1+\bar{\gamma}_d)\bar{c}(1+a)\frac{\partial\bar{\theta}}{\partial y}\delta_2 - Ra_L^{-\frac{1}{5}(1+\omega^2)}[r\bar{c} + (1+\bar{\gamma}_d)\Delta][\frac{\bar{u}(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{\bar{x}}}$$

$$+ \frac{\bar{u}\theta}{(1+Aqy)} \frac{da}{d\bar{x}} + \bar{v}(1+a)\frac{\partial\bar{\theta}}{\partial y}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+a)T[\frac{\partial^2\bar{\theta}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial y}$$

$$\pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)} \frac{\partial\bar{\theta}}{\partial y}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+a)\frac{\partial\bar{\theta}}{\partial y}\frac{dT}{dy} = 0$$

The above set of equations can be employed in the stability analysis of any free-convection flow which satisfies the boundary-layer equations 2.1-3 and the boundary conditions 2.1-4, assuming that the disturbance has the form of Tollmien-Schlichting waves. However, auxiliary relationships for r, Λ, Δ and T and a set of boundary conditions must be established before a solution can be attempted.

By considering constant-property flows over two-dimensional surfaces, equations 2.3-3 can be replaced by a set of more simplified equations. From the Boussinesq approximation,

$$\rho = \rho_\infty(1-\beta\Theta).$$

However, since $\rho = \bar{\rho} + \rho^*$

and $\Theta = \bar{\Theta} + \Theta^*$,

then $\bar{\rho} = \rho_\infty(1-\beta\bar{\Theta})$

and $\rho^* = - \rho_\infty\beta\Theta^*$.

Using the normalized forms of these expressions along with the above assumptions and assuming $\varepsilon<<1$ where $\varepsilon = \beta\Theta_w$, the disturbance continuity equation reduces to a form which can be satisfied by introducing a disturbance stream function, ψ^*, defined by:

$$u^* = \frac{\partial\psi^*}{\partial y}$$

and

$$v^* = \frac{-1}{(1+Aqy)}\frac{\partial\psi^*}{\partial x^*} .$$

Utilizing the above assumptions and introducing the disturbance stream function, equations 2.3-2 reduce to the following:

$$\frac{1}{(1+Aqy)^2}\frac{\partial^3\psi^*}{\partial x^{*2}\partial\tau} - \frac{AyRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)^3}\frac{dq}{d\bar{x}}\frac{\partial^2\psi^*}{\partial x^*\partial\tau} + \frac{\partial^3\psi^*}{\partial y^2\partial\tau} + \frac{Aq}{(1+Aqy)}\frac{\partial^2\psi^*}{\partial y\partial\tau}$$

$$+ \frac{\bar{u}}{(1+Aqy)^3}\left[\frac{\partial^3\psi^*}{\partial x^{*3}} - \frac{2AyRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{dq}{d\bar{x}}\frac{\partial^2\psi^*}{\partial x^{*2}}\right] - \frac{Aq\bar{u}}{(1+Aqy)^2}\frac{\partial^2\psi^*}{\partial y\partial x^*}$$

$$+ \frac{\bar{u}}{(1+Aqy)}\frac{\partial^3\psi^*}{\partial x^*\partial y^2} + \frac{A^2q^2\bar{u}}{(1+Aqy)^3}\frac{\partial\psi^*}{\partial x^*} - \frac{Aq}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial y}\frac{\partial\psi^*}{\partial x^*} - \frac{1}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y^2}\frac{\partial\psi^*}{\partial x^*}$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{1}{(1+Aqy)^3}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial^2\psi^*}{\partial x^{*2}} - \frac{Ay\bar{u}}{(1+Aqy)^4}\frac{dq}{d\bar{x}}\frac{\partial^2\psi^*}{\partial x^{*2}} + \frac{\bar{v}}{(1+Aqy)^2}\frac{\partial^3\psi^*}{\partial x^{*2}\partial y}\right.$$

$$+ \frac{2Aq}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\psi^*}{\partial y} + \frac{2A\bar{u}}{(1+Aqy)^2}\frac{dq}{d\bar{x}}\frac{\partial\psi^*}{\partial y} - \frac{2A^2qy\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}}\frac{\partial\psi^*}{\partial y}$$

$$+ \frac{1}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}}\frac{\partial\psi^*}{\partial y} + \frac{\partial\bar{v}}{\partial y}\frac{\partial^2\psi^*}{\partial y^2} + \frac{2Aq\bar{v}}{(1+Aqy)}\frac{\partial^2\psi^*}{\partial y^2} + \frac{\bar{v}\partial^3\psi^*}{\partial y^3}$$

$$\left.+ \frac{1}{(1+Aqy)^2}\frac{\partial\bar{v}}{\partial y}\frac{\partial^2\psi^*}{\partial x^{*2}} - \frac{Aq\bar{v}}{(1+Aqy)^3}\frac{\partial^2\psi^*}{\partial x^{*2}} + \frac{1}{(1+Aqy)}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial^2\psi^*}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{v}}{\partial y}\frac{\partial\psi^*}{\partial y}\right]$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{A^3q^3}{(1+Aqy)^3}\frac{\partial\psi^*}{\partial y} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial^2\psi^*}{\partial y^2} + \frac{2Aq}{(1+Aqy)}\frac{\partial^3\psi^*}{\partial y^3}\right.$$

$$+ \frac{3A^2q^2}{(1+Aqy)^4}\frac{\partial^2\psi^*}{\partial x^{*2}} - \frac{2Aq}{(1+Aqy)^3}\frac{\partial^3\psi^*}{\partial x^{*2}\partial y} + \frac{1}{(1+Aqy)^4}\frac{\partial^4\psi^*}{\partial x^{*4}} + \frac{2}{(1+Aqy)^2}\frac{\partial^4\psi^*}{\partial x^{*2}\partial y^2}$$

[illegible]

$$+ \frac{\partial^4\psi^*}{\partial y^4}] + \sigma Ra_L^{-\omega^2} [\frac{Aq\sin\alpha\ \theta^*}{(1+Aqy)} + Ra_L^{-\frac{1}{5}(1+\omega^2)} \frac{\sin\alpha}{(1+Aqy)} \frac{d\alpha\theta^*}{d\bar{x}}$$

$$- \frac{1}{(1+Aqy)} \frac{\partial\theta^*}{\partial x^*} \cos\alpha + \frac{\partial\theta^*}{\partial y} \sin\alpha]$$

(2.3-4)

$$(1+a)\frac{\partial\theta^*}{\partial\tau} + \frac{(1+a)\bar{u}}{(1+Aqy)} \frac{\partial\theta^*}{\partial x^*} - \frac{(1+a)}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial y} \frac{\partial\psi^*}{\partial x^*} + Ra_L^{-\frac{1}{5}(1+\omega^2)} [\frac{\bar{u}\theta^*}{(1+Aqy)} \frac{da}{d\bar{x}}$$

$$+ \frac{(1+a)\bar{v}\partial\theta^*}{\partial y} + \frac{(1+a)}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial\bar{x}} \frac{\partial\psi^*}{\partial y} + \frac{\bar{\theta}}{(1+Aqy)} \frac{da}{d\bar{x}} \frac{\partial\psi^*}{\partial y}]$$

$$= Ra_L^{-\frac{1}{5}(1+\omega^2)} (1+a) [\frac{1}{(1+Aqy)^2} \frac{\partial^2\theta^*}{\partial x^{*2}} + \frac{\partial^2\theta^*}{\partial y^2} + \frac{Aq}{(1+Aqy)} \frac{\partial\theta^*}{\partial y}]$$

Following the procedure used to obtain equations 2.3-1, it is assumed that ψ^* and θ^* are proportional to exp $[i(\lambda x^* - B\tau)]$ and a single component of a Fourier series in x^* is used to analyse the stability of the flow. Therefore, ψ^* and θ^* are assumed to have the following forms:

$$\psi^*(x^*, y, \tau) = \phi(y)\ e^{i(\lambda x^* - B\tau)}$$

and $$\theta^*(x^*, y, \tau) = s(y)\ e^{i(\lambda x^* - B\tau)}$$

Substituting these expressions into equations 2.3-4 and eliminating the exponential factor in each term, the set of disturbance equations becomes:

$$\{\frac{i\lambda}{(1+Aqy)}[\frac{\lambda}{(1+Aqy)}(B - \frac{\lambda\bar{u}}{(1+Aqy)}) - \frac{\partial^2\bar{u}}{\partial y^2} + \frac{A^2q^2\bar{u}}{(1+Aqy)^2} - \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y}]$$

$$- \frac{\lambda Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)^3}[AyB\frac{dq}{d\bar{\bar{x}}} - \frac{3Ay\lambda\bar{u}}{(1+Aqy)}\frac{dq}{d\bar{\bar{x}}} + \lambda\frac{\partial\bar{u}}{\partial\bar{x}} - Aq\lambda\bar{v} + \lambda(1+Aqy)\frac{\partial\bar{v}}{\partial y}]\}\ \phi$$

$$+ \{-i[\frac{Aq\bar{u}\lambda}{(1+Aqy)^2} + \frac{BAq}{(1+Aqy)}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{2Aq}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial\bar{x}} - \frac{\lambda^2\bar{v}}{(1+Aqy)^2}$$

$$+ \frac{2A\bar{u}}{(1+Aqy)^2}\frac{dq}{d\bar{\bar{x}}} - \frac{2A^2qy\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{\bar{x}}} + \frac{1}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{v}}{\partial y}]\}\ \frac{d\phi}{dy}$$

$$- \{i[B - \frac{\lambda\bar{u}}{(1+Aqy)}] - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\partial\bar{v}}{\partial y} + \frac{2Aq\bar{v}}{(1+Aqy)} + \frac{1}{(1+Aqy)}\frac{\partial\bar{u}}{\partial\bar{x}}]\}\ \frac{d^2\phi}{dy^2}$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{d^3\phi}{dy^3} + \sigma Ra_L^{-\omega^2}\{\frac{s}{(1+Aqy)}[i\lambda\cos\alpha - Aq\sin\alpha$$

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}\sin\alpha\frac{d\alpha}{d\bar{\bar{x}}}] - \frac{ds}{dy}\sin\alpha\}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{d^4\phi}{dy^4} + \frac{2Aq}{(1+Aqy)}\frac{d^3\phi}{dy^3} - [\frac{2\lambda^2}{(1+Aqy)^2} + \frac{A^2q^2}{(1+Aqy)^2}]\frac{d^2\phi}{dy^2}$$

$$+ \frac{1}{(1+Aqy)^3}(A^3q^3 + 2\lambda^2Aq)\frac{d\phi}{dy} + \frac{\lambda^2}{(1+Aqy)^4}(\lambda^2 - 3A^2q^2)\phi\} \tag{2.3-5}$$

$$\{(1+a)[B - \frac{\lambda\bar{u}}{(1+Aqy)}] + iRa_L^{-\frac{1}{5}(1+\omega^2)} \frac{\bar{u}}{(1+Aqy)} \frac{da}{d\bar{x}}\} s + iRa_L^{-\frac{1}{5}(1+\omega^2)}(1+a)\bar{v}\frac{ds}{dy}$$

$$+ \frac{\lambda(1+a)}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial y} \phi + \frac{iRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)} [(1+a)\frac{\partial\bar{\theta}}{\partial\bar{x}} + \bar{\theta}\frac{da}{d\bar{x}}]\frac{d\phi}{dy}$$

$$= iRa_L^{-\frac{1}{5}(1+\omega^2)}(1+a)[\frac{d^2s}{dy^2} + \frac{Aq}{(1+Aqy)} \frac{ds}{dy} - \frac{\lambda^2}{(1+Aqy)^2} s]$$

It is possible to simplify the disturbance equations further by assuming that the mean flow is parallel. This assumption implies that the mean-flow velocities and temperature do not vary with $\bar{x}$ and that $\bar{v}$ is approximately zero. Utilizing this assumption, equations 2.3-5 reduce to:

$$\{\frac{i\lambda}{(1+Aqy)}[\frac{\lambda}{(1+Aqy)}(B - \frac{\lambda\bar{u}}{(1+Aqy)}) - \frac{\partial^2\bar{u}}{\partial y^2} - \frac{A^2q^2\bar{u}}{(1+Aqy)^2} - \frac{2Aq}{(1+Aqy)} \frac{\partial\bar{u}}{\partial y}]$$

$$- \frac{\lambda Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)^3} [AyB\frac{dq}{d\bar{x}} - \frac{3Ay\lambda\bar{u}}{(1+Aqy)} \frac{dq}{d\bar{x}}]\}\phi + \{\frac{-iAq}{(1+Aqy)}[\frac{\lambda\bar{u}}{(1+Aqy)} + B]$$

$$+ \frac{2A\bar{u}Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)^2} [1 - \frac{Aqy}{(1+Aqy)}]\frac{dq}{d\bar{x}}\} \frac{d\phi}{dy} - i[B - \frac{\lambda\bar{u}}{(1+Aqy)}]\frac{d^2\phi}{dy^2}$$

$$+ \sigma Ra_L^{-\omega^2} \{\frac{s}{(1+Aqy)}[i\lambda\cos\alpha - Aq\sin\alpha - Ra_L^{-\frac{1}{5}(1+\omega^2)}\sin\alpha\frac{d\alpha}{d\bar{x}}] - \frac{ds}{dy} \sin\alpha\}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)} \{\frac{d^4\phi}{dy^4} + \frac{2Aq}{(1+Aqy)} \frac{d^3\phi}{dy^3} - \frac{1}{(1+Aqy)^2}(2\lambda^2 + A^2q^2)\frac{d^2\phi}{dy^2}$$

$$+ \frac{Aq}{(1+Aqy)^3}(A^2q^2 + 2\lambda^2)\frac{d\phi}{dy} + \frac{\lambda^2}{(1+Aqy)^4}(\lambda^2 - 3A^2q^2)\ \phi\}$$

(2.3-6)

$$\{(1+a)[B - \frac{\lambda\bar{u}}{(1+Aqy)}] + \frac{i\bar{u}Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{da}{d\bar{x}}\}\ s + \frac{\lambda(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}\phi}{\partial y} + \frac{i\bar{\theta}Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{da}{d\bar{x}}\frac{d\phi}{dy}$$

$$= iRa_L^{-\frac{1}{5}(1+\omega^2)}(1+a)[\frac{d^2s}{dy^2} + \frac{Aq}{(1+Aqy)}\frac{ds}{dy} - \frac{\lambda^2}{(1+Aqy)^2}\ s]$$

Before attempting to solve either equations 2.3-5 or 2.3-6, a set of boundary conditions must be established. Firstly, from the assumption of no flow along or through the surface the disturbance velocities vanish along the surface y = 0 and far from the surface. Secondly, the disturbance temperature is assumed to vanish far from the surface. These conditions can be stated as follows:

$$y = 0 \quad : \quad \phi = \frac{d\phi}{dy} = 0$$

(2.3-7)

and

$$y \to \infty \quad : \quad \phi = \frac{d\phi}{dy} = s = 0$$

The remaining condition on s is not easily specified. As Knowles and Gebhart [36] pointed out, the second condition for s depends on the thermal capacity and the thermal conductance of the material used for the heated surface. If the thermal capacity of the surface material and the thermal conductance normal to the surface are large, it is possible to maintain an isothermal surface, implying that the disturbance temperature is zero at the surface. If the thermal capacity of the surface material is zero and the thermal conductance is negligible in the direction parallel to y = 0,

it is possible to maintain a uniform heat flux along the surface, implying that the disturbance heat flux is zero at the surface. The intermediate conditions can be obtained by considering an energy balance on a small element of the material between the heaters, used for heating the surface, and the boundary, y = 0. Consider the energy balance after subtracting the mean-flow energy terms and neglecting the disturbance conduction terms in the direction parallel to y = 0, that is,

$$\rho b\, C_p \frac{\partial \Theta^*}{\partial t}\Big|_{Y=0} - k \frac{\partial \Theta^*}{\partial Y}\Big|_{Y=0} \simeq 0$$

or

$$s(o) = \frac{iL}{Bb} Ra_L^{-\frac{2}{5}(1+\omega^2)} \frac{ds(o)}{dy}, \tag{2.3-8}$$

where b is the thickness of the surface material and it is assumed to be small relative to L. The above conditions 2.3-7 and 2.3-8 are applicable to either equations 2.3-5 or 2.3-6.

2.3-4 Taylor-Goertler Roll-Vortex Disturbances

The roll-vortex disturbances described in section 2.3-2 are called Taylor-Goertler roll vortices as are their counterparts in forced-convection flows. These roll vortices are independent of x*, and therefore, the set of equations D-1 can be simplified to the following:

$$\frac{\partial \gamma_d^*}{\partial \tau} + (1+\bar{\gamma}_d)\left[\frac{\partial v^*}{\partial y} + \frac{Aqv^*}{(1+Aqy)} + \frac{1}{(j \pm Jy\sin\alpha)}\left(\frac{\partial w}{\partial z} \pm v^* J\sin\alpha\right)\right] + \frac{v^* \partial \bar{\gamma}_d}{\partial y}$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}\left\{\frac{\pm\gamma_d^* \bar{v} J\sin\alpha}{(j \pm Jy\sin\alpha)} + \frac{1}{(j \pm Jy\sin\alpha)}\left(\frac{dj}{d\bar{x}} \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}}\right)\left[\frac{(1+\bar{\gamma}_d)u^*}{(1+Aqy)}\right.\right.$$

$$+ \frac{\gamma_d{}^*\bar{u}}{(1+Aqy)}] + \frac{1}{(1+Aqy)}[\gamma_d{}^*(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}) + u^*\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}] + \gamma_d{}^*\frac{\partial\bar{v}}{\partial y} + \frac{\bar{v}\partial\gamma_d{}^*}{\partial y}\} = 0$$

$$(1+\bar{\gamma}_d)\{\frac{\partial^2 w}{\partial y\partial\tau} + \frac{1}{(j\pm Jysin\alpha)}[\frac{2Aq\bar{u}}{(1+Aqy)}\frac{\partial u^*}{\partial z} \pm Jsin\alpha\frac{\partial w}{\partial\tau} - \frac{\partial^2 v^*}{\partial z\partial\tau}]\} + \frac{\partial\bar{\gamma}_d}{\partial y}\frac{\partial w}{\partial\tau}$$

$$+ \frac{Aq\bar{u}^2}{(1+Aqy)(j\pm Jysin\alpha)}\frac{\partial\gamma_d{}^*}{\partial z} + Ra_L{}^{-\frac{1}{5}(1+\omega^2)}\{(1+\bar{\gamma}_d)(\frac{\bar{v}\partial^2 w}{\partial y^2} + \frac{\partial\bar{v}}{\partial y}\frac{\partial w}{\partial y})$$

$$+ \frac{(1+\bar{\gamma}_d)}{(j\pm Jysin\alpha)}[\frac{1}{(1+Aqy)}(\frac{dj}{d\bar{x}} \pm Jycos\alpha\frac{d\alpha}{d\bar{x}})(\frac{w\partial\bar{u}}{\partial y} + \frac{\bar{u}\partial w}{\partial y} - \frac{Aqw\bar{u}}{(1+Aqy)}) - \frac{\bar{v}\partial^2 v^*}{\partial z\partial y}$$

$$\pm 2\bar{v}Jsin\alpha\frac{\partial w}{\partial y} \pm \frac{w\bar{u}Jcos\alpha}{(1+Aqy)}\frac{d\alpha}{d\bar{x}} - \frac{\partial\bar{v}}{\partial y}\frac{\partial v^*}{\partial z} \pm wJsin\alpha\frac{\partial\bar{v}}{\partial y}] + \frac{\bar{v}\partial\bar{\gamma}_d}{\partial y}\frac{\partial w}{\partial y}\}$$

$$= \sigma Ra_L{}^{-\frac{1}{5}(1+\omega^2)}\{\frac{\partial^2\bar{M}}{\partial y^2}[\frac{\partial w}{\partial y} + \frac{1}{(j\pm Jysin\alpha)}(\frac{\partial v^*}{\partial z} \mp wJsin\alpha)] + \frac{\partial\bar{M}}{\partial y}[\frac{2\partial^2 w}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial w}{\partial y} \mp \frac{Jsin\alpha}{(j\pm Jysin\alpha)}\frac{\partial w}{\partial y} + \frac{1}{(1+Aqy)(j\pm Jysin\alpha)}(\frac{Aq\partial v^*}{\partial z} \pm AqwJsin\alpha)$$

$$+ \frac{1}{(j\pm Jysin\alpha)^2}(\frac{2\partial^2 w}{\partial z^2} \pm 3Jsin\alpha\frac{\partial v^*}{\partial z} - wJ^2 sin^2\alpha)] + \bar{M}[\frac{\partial^3 w}{\partial y^3} + \frac{Aq}{(1+Aqy)}\frac{\partial^2 w}{\partial y^2}$$

$$- \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial w}{\partial y}] + \frac{\bar{M}}{(j\pm Jysin\alpha)}[\frac{\partial^3 w}{\partial z^2\partial y} \pm 2Jsin\alpha\frac{\partial^2 w}{\partial y^2} - \frac{\partial^3 v^*}{\partial y^2\partial z} + \frac{Aq}{(1+Aqy)}($$

$$\pm Jsin\alpha\frac{\partial w}{\partial y} - \frac{\partial^2 v^*}{\partial y\partial z}) + \frac{A^2q^2}{(1+Aqy)^2}(\frac{\partial v^*}{\partial z} \mp wJsin\alpha)] + \frac{\bar{M}}{(j\pm Jysin\alpha)^2}($$

$$\pm \frac{J\sin\alpha\partial^2 v^*}{\partial y\partial z} - \frac{J^2\sin^2\alpha\partial w}{\partial y} \pm \frac{J\sin\alpha\partial^2 w}{\partial z^2} - \frac{\partial^3 v^*}{\partial z^3}) + \frac{\bar{M}J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^3}(\pm wJ\sin\alpha$$

$$- \frac{\partial v^*}{\partial z})\} + \frac{\sigma\cos\alpha}{\beta\bar{\Theta}_c} Ra_L^{-\omega^2}\frac{\partial\gamma_d{}^*}{\partial z}$$

$$\frac{(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 u^*}{\partial z\partial\tau} + \frac{\partial\bar{u}}{\partial y}\frac{\partial v^*}{\partial z} + \frac{Aq\bar{u}}{(1+Aqy)}\frac{\partial v^*}{\partial z}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}[\frac{\bar{v}\partial^2 u^*}{\partial z\partial y}$$

$$+ \frac{1}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial u^*}{\partial z} + \frac{Aq\bar{v}\partial u^*}{\partial z}) - \frac{1}{(1+Aqy)}(\frac{dj}{d\bar{x}} \pm \frac{Jy\cos\alpha d\alpha}{d\bar{x}})\frac{\partial w}{\partial\tau}]$$

$$- \frac{1}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}\frac{\partial w}{\partial\tau} + \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial\gamma_d{}^*}{\partial z}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial^2 M^*}{\partial z\partial y}[\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}] + \frac{1}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 u^*}{\partial z\partial y}$$

$$- \frac{Aq}{(1+Aqy)}\frac{\partial u^*}{\partial z}]\frac{\partial\bar{M}}{\partial y} + \frac{\bar{M}}{(j\pm Jy\sin\alpha)}[\frac{\partial^3 u^*}{\partial z\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial^2 u^*}{\partial z\partial y} - \frac{1}{(1+Aqy)^2}(\frac{A^2q^2\partial u^*}{\partial z}$$

$$\pm \frac{AwJ\sin\alpha dq}{d\bar{x}})] + \frac{\bar{M}}{(j\pm Jy\sin\alpha)^2}[\pm \frac{J\sin\alpha\partial^2 u^*}{\partial z\partial y} \pm \frac{AqJ\sin\alpha}{(1+Aqy)}\frac{\partial u^*}{\partial z}]$$

$$+ \frac{\bar{M}}{(j\pm Jy\sin\alpha)^3}\frac{\partial^3 u^*}{\partial z^3} + \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial M^*}{\partial z}[\frac{\partial^2\bar{u}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q^2\bar{u}}{(1+Aqy)^2}]$$

$$\pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)^2}\frac{\partial M^*}{\partial z}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\} - \frac{\sigma\sin\alpha}{\beta\bar{\Theta}_c} Ra_L^{-\omega^2}\frac{\partial\gamma_d{}^*}{\partial z}$$

(2.3-9)

$$-(1+\bar{\gamma}_d)[\frac{Aq}{(1+Aqy)}(\frac{\partial u^*}{\partial\tau} + \frac{2v^*\partial\bar{u}}{\partial y} + \frac{\bar{u}\partial v^*}{\partial y}) + \frac{\partial^2 u^*}{\partial y\partial\tau} + \frac{v^*\partial^2\bar{u}}{\partial y^2} + \frac{\partial v^*}{\partial y}\frac{\partial\bar{u}}{\partial y}] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\partial u^*}{\partial\tau}$$

$$+ \frac{v^*\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}v^*}{(1+Aqy)}] + Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{1}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}[\frac{\partial v^*}{\partial\tau} - \frac{2Aq\bar{u}u^*}{(1+Aqy)}]$$

$$-(1+\bar{\gamma}_d)[\frac{\bar{v}\partial^2 u^*}{\partial y^2} + \frac{\partial\bar{v}}{\partial y}\frac{\partial u^*}{\partial y} + \frac{1}{(1+Aqy)}(\frac{2Aq\bar{v}\partial u^*}{\partial y} + u^*\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial u^*}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}}$$

$$+ \frac{Aqu^*\partial\bar{v}}{\partial y}) + \frac{2Au^*}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}} + \frac{q\partial\bar{u}}{\partial\bar{x}}) - \frac{2A^2qy\bar{u}u^*}{(1+Aqy)^3}\frac{dq}{d\bar{x}}] - \gamma_d^*[\frac{\bar{v}\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aq\bar{u}\partial\bar{v}}{\partial y} + \frac{2Aq\bar{v}\partial\bar{u}}{\partial y}) + \frac{A\bar{u}}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}}$$

$$+ \frac{2q\partial\bar{u}}{\partial\bar{x}}) - \frac{A^2qy\bar{u}^2}{(1+Aqy)^3}\frac{dq}{d\bar{x}}] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\bar{v}\partial u^*}{\partial y} + \frac{u^*}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]$$

$$- \frac{\partial\gamma_d^*}{\partial y}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{\partial^2\bar{M}}{\partial y^2}[\frac{Aqu^*}{(1+Aqy)} - \frac{\partial u^*}{\partial y}] - \frac{\partial\bar{M}}{\partial y}[\frac{2\partial^2 u^*}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial u^*}{\partial y}$$

$$- \frac{A^2q^2u^*}{(1+Aqy)^2}] - \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial\bar{M}}{\partial y}[\pm\frac{Aqu^*J\sin\alpha}{(1+Aqy)} \pm \frac{J\sin\alpha\partial u^*}{\partial y}$$

$$+ \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial^2 u^*}{\partial z^2}] + \frac{\partial^2 M^*}{\partial y^2}[\frac{Aq\bar{u}}{(1+Aqy)} - \frac{\partial\bar{u}}{\partial y}] - \frac{\partial M^*}{\partial y}[\frac{2\partial^2\bar{u}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y}$$

$$- \frac{A^2q\bar{u}}{(1+Aqy)^2}] \mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)} \frac{\partial M^*}{\partial y}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}] - \bar{M}[\frac{\partial^3u^*}{\partial y^3} + \frac{2Aq}{(1+Aqy)} \frac{\partial^2u^*}{\partial y^2}$$

$$- \frac{A^2q^2}{(1+Aqy)^2} \frac{\partial u^*}{\partial y} + \frac{A^3q^3u^*}{(1+Aqy)^3}] \mp \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2u^*}{\partial y^2} + \frac{2Aq}{(1+Aqy)} \frac{\partial u^*}{\partial y}]$$

$$+ \frac{\bar{M}}{(j\pm Jy\sin\alpha)^2}[J^2\sin^2\alpha\frac{\partial u^*}{\partial y} - \frac{\partial^3u^*}{\partial y\partial z^2} + \frac{Aq}{(1+Aqy)}(u^*J^2\sin^2\alpha - \frac{\partial^2u^*}{\partial z^2})]$$

$$\pm \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)^3} \frac{\partial^2u^*}{\partial z^2} - M^*[\frac{\partial^3\bar{u}}{\partial y^3} + \frac{2Aq}{(1+Aqy)} \frac{\partial^2\bar{u}}{\partial y^2} - \frac{A^2q^2}{(1+Aqy)^2} \frac{\partial\bar{u}}{\partial y} + \frac{A^3q^3\bar{u}}{(1+Aqy)^3}]$$

$$\mp \frac{M^*J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2\bar{u}}{\partial y^2} + \frac{2Aq}{(1+Aqy)} \frac{\partial\bar{u}}{\partial y}] + \frac{M^*J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\}$$

$$+ \frac{\sigma Ra_L^{-\omega^2}}{\beta\bar{\Theta}_c}[\frac{\partial\gamma_d^*}{\partial y}\sin\alpha + \frac{\gamma_d^*Aq\sin\alpha}{(1+Aqy)} + Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\gamma_d^*\sin\alpha}{(1+Aqy)} \frac{d\alpha}{d\bar{x}}]$$

$$(1+\bar{\gamma}_d)\bar{c}(1+a)(\frac{\partial\bar{\theta}^*}{\partial\tau} + v^*\frac{\partial\bar{\theta}}{\partial y}) + Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+\bar{\gamma}_d)\bar{c}[\frac{1}{(1+Aqy)} \frac{da}{d\bar{x}}(\bar{u}\theta^* + u^*\bar{\theta})$$

$$+ \bar{v}(1+a)\frac{\partial\theta^*}{\partial y} + \frac{u^*(1+a)}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial\bar{x}}] + [\frac{\bar{u}(1+a)}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial\bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)} \frac{da}{d\bar{x}}$$

$$+ \bar{v}(1+a)\frac{\partial\bar{\theta}}{\partial y}][\gamma_d^*\bar{c} + (1+\bar{\gamma}_d)c^*]\}$$

$$= (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\{K^*[\frac{\partial^2\bar{\theta}}{\partial y^2} + \frac{Aq}{(1+Aqy)} \frac{\partial\bar{\theta}}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)} \frac{\partial\bar{\theta}}{\partial y}]$$

$$+ \bar{K}\left[\frac{\partial^2\theta^*}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\theta^*}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\theta^*}{\partial y} + \frac{1}{(j\pm Jy\sin\alpha)^2}\frac{\partial^2\theta^*}{\partial z^2}\right]$$

$$+ \frac{\partial K^*}{\partial y}\frac{\partial\bar{\theta}}{\partial y} + \frac{\partial\bar{K}}{\partial y}\frac{\partial\theta^*}{\partial y}\}$$

An approach similar to that employed in section 2.3-3 is used in the analysis of the above equations. If it is assumed that the disturbance variables are proportional to exp $[i(\Omega z - B\tau)]$ and that the stability of the flow can be analysed by considering a single component of a Fourier series in z, the disturbance variables may be written as follows:

$$u^*(y, z, \tau) = \delta_1(y)\, e^{i(\Omega z - B\tau)}$$

$$v^*(y, z, \tau) = \delta_2(y)\, e^{i(\Omega z - B\tau)}$$

$$w(y, z\ \tau) = \delta_3(y)\, e^{i(\Omega z - B\tau)}$$

$$\theta^*(y, z, \tau) = s(y)\, e^{i(\Omega z - B\tau)}$$

$$\gamma_d^*(y, z, \tau) = r(y)\, e^{i(\Omega z - B\tau)}$$

$$M^*(y, z, \tau) = \Lambda(y)\, e^{i(\Omega z - B\tau)}$$

$$c^*(y, z, \tau) = \Delta(y)\, e^{i(\Omega z - B\tau)}$$

$$K^*(y, z, \tau) = T(y)\, e^{i(\Omega z - B\tau)}$$

Substituting these expressions into equations 2.3-9 and eliminating the exponential factor in each term leads to the following set of equations:

$$(1+\bar{\gamma}_d)\frac{d\delta_2}{dy} + \left\{\frac{\partial\bar{\gamma}_d}{\partial y} + (1+\bar{\gamma}_d)\left[\frac{Aq}{(1+Aqy)} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\right]\right\}\delta_2 + \frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\left[\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}\right.$$

$$\left. + \frac{(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}\left(\frac{dj}{d\bar{x}} \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}}\right)\right]\delta_1 + \frac{i\Omega(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}\delta_3 + Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{dr}{dy}$$

$$+ \left\{Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{1}{(1+Aqy)}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}\right) + \frac{\partial\bar{v}}{\partial y} + \frac{\bar{u}}{(1+Aqy)(j\pm Jy\sin\alpha)}\left(\frac{dj}{d\bar{x}}\right.\right.\right.$$

$$\left.\left.\left. \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}}\right) \pm \frac{\bar{v}J\sin\alpha}{(j\pm Jy\sin\alpha)}\right] - iB\right\} r = 0$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+\bar{\gamma}_d)\bar{v}\frac{d^2\delta_3}{dy^2} + \left\{(1+\bar{\gamma}_d)Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{\partial\bar{v}}{\partial y} + \frac{\bar{u}}{(1+Aqy)(j\pm Jy\sin\alpha)}\left(\frac{dj}{d\bar{x}}\right.\right.\right.$$

$$\left.\left.\left. \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}}\right) \pm \frac{2\bar{v}J\sin\alpha}{(j\pm Jy\sin\alpha)}\right] + Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{\partial\bar{\gamma}_d}{\partial y} - iB(1+\bar{\gamma}_d)\right\}\frac{d\delta_3}{dy}$$

$$+ \left\{-iB\frac{\partial\bar{\gamma}_d}{\partial y} \mp \frac{iB(1+\bar{\gamma}_d)J\sin\alpha}{(j\pm Jy\sin\alpha)} + \frac{(1+\bar{\gamma}_d)Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(j\pm Jy\sin\alpha)}\left[\frac{1}{(1+Aqy)}\left(\frac{dj}{d\bar{x}}\right.\right.\right.$$

$$\left.\left.\left. \pm Jy\cos\alpha\frac{d\alpha}{d\bar{x}}\right)\left(\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}\right) \pm \frac{\bar{u}J\cos\alpha}{(1+Aqy)}\frac{d\alpha}{d\bar{x}} \pm J\sin\alpha\frac{\partial\bar{v}}{\partial y}\right]\right\}\delta_3$$

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{i\Omega\bar{v}(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}\frac{d\delta_2}{dy} - \frac{(1+\bar{\gamma}_d)\Omega}{(j\pm Jy\sin\alpha)}\left[B + iRa_L^{-\frac{1}{5}(1+\omega^2)}\frac{\partial\bar{v}}{\partial y}\right]\delta_2$$

$$+ \frac{2i\Omega Aq(1+\bar{Y}_d)\bar{u}\delta_1}{(1+Aqy)(j\pm Jy\sin\alpha)} + \frac{i\Omega Aq\bar{u}^2 r}{(1+Aqy)(j\pm Jy\sin\alpha)}$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{M}\frac{d^3\delta_3}{dy^3} + \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{2\partial\bar{M}}{\partial y} + \frac{\bar{M}Aq}{(1+Aqy)} \pm \frac{2\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}\right]\frac{d^2\delta_3}{dy^2}$$

$$+ \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left\{\frac{\partial^2\bar{M}}{\partial y^2} + \frac{\partial\bar{M}}{\partial y}\left[\frac{Aq}{(1+Aqy)} \mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\right] - \frac{\bar{M}A^2q^2}{(1+Aqy)^2}\right.$$

$$\left. - \frac{\bar{M}}{(j\pm Jy\sin\alpha)}\left[\Omega^2 \mp \frac{AqJ\sin\alpha}{(1+Aqy)}\right] - \frac{\bar{M}J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}\right\}\frac{d\delta_3}{dy}$$

$$- \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left\{\frac{\partial\bar{M}}{\partial y}\left[\frac{1}{(j\pm Jy\sin\alpha)^2}(2\Omega^2 + J^2\sin^2\alpha) \mp \frac{AqJ\sin\alpha}{(1+Aqy)(j\pm Jy\sin\alpha)}\right]\right.$$

$$\pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial^2\bar{M}}{\partial y^2} \pm \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}\left[\frac{A^2q^2}{(1+Aqy)^2} + \frac{\Omega^2}{(j\pm Jy\sin\alpha)}\right.$$

$$\left.\left. - \frac{\bar{M}J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}\right]\right\}\delta_3 - \frac{i\Omega\bar{M}\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(j\pm Jy\sin\alpha)}\frac{d^2\delta_2}{dy^2} - \frac{i\Omega\bar{M}\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(j\pm Jy\sin\alpha)}\left[\frac{Aq}{(1+Aqy)}\right.$$

$$\left. \mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\right]\frac{d\delta_2}{dy} + \frac{i\Omega\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(j\pm Jy\sin\alpha)}\left\{\frac{\partial^2\bar{M}}{\partial y^2} + \frac{\partial\bar{M}}{\partial y}\left[\frac{Aq}{(1+Aqy)} \pm \frac{3J\sin\alpha}{(j\pm Jy\sin\alpha)}\right]\right.$$

$$\left. + \frac{\bar{M}A^2q^2}{(1+Aqy)^2} + \frac{\Omega^2\bar{M}}{(j\pm Jy\sin\alpha)} - \frac{\bar{M}J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}\right\}\delta_2 + \frac{i\Omega\sigma\cos\alpha}{\beta\bar{\Theta}_c} Ra_L^{-\omega^2} r$$

$$iRa_L^{-\frac{1}{5}(1+\omega^2)}\frac{(1+\bar{Y}_d)\Omega\bar{v}}{(j\pm Jy\sin\alpha)}\frac{d\delta_1}{dy} + \frac{\Omega(1+\bar{Y}_d)}{(j\pm Jy\sin\alpha)}\left[B + \frac{iRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}\right)\right]\delta_1$$

$$+ \frac{i\Omega(1+\bar{\gamma}_d)}{(j\pm Jysin\alpha)}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\delta_2 + \frac{iBRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}[\frac{\partial\bar{\gamma}_d}{\partial\bar{x}} + \frac{(1+\bar{\gamma}_d)}{(j\pm Jysin\alpha)}(\frac{dj}{d\bar{x}}$$

$$\pm Jycos\alpha\frac{d\alpha}{d\bar{x}})]\delta_3 + \frac{i\Omega Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(j\pm Jysin\alpha)}[\bar{v}\frac{\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\ r$$

$$= \frac{i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{M}\Omega}{(j\pm Jysin\alpha)}\frac{d^2\delta_1}{\partial y^2} + \frac{i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\Omega}{(j\pm Jysin\alpha)}[\frac{\partial\bar{M}}{\partial y} + \frac{\bar{M}Aq}{(1+Aqy)}$$

$$\pm \frac{\bar{M}Jsin\alpha}{(j\pm Jysin\alpha)}]\frac{d\delta_1}{dy} - \frac{i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\Omega}{(j\pm Jysin\alpha)}[\frac{Aq}{(1+Aqy)}\frac{\partial\bar{M}}{\partial y} + \frac{\bar{M}A^2q^2}{(1+Aqy)^2}$$

$$\pm \frac{\bar{M}AqJsin\alpha}{(1+Aqy)(j\pm Jysin\alpha)} + \frac{\bar{M}\Omega^2}{(j\pm Jysin\alpha)^2}]\delta_1 \mp \frac{\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{M}AJsin\alpha}{(1+Aqy)^2(j\pm Jysin\alpha)}\frac{dq}{d\bar{x}}\delta_3$$

$$+ \frac{i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\Omega}{(j\pm Jysin\alpha)}[\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}]\frac{d\Lambda}{dy} + \frac{i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\Omega}{(j\pm Jysin\alpha)}\{\frac{\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q^2\bar{u}}{(1+Aqy)^2} \pm \frac{Jsin\alpha}{(j\pm Jysin\alpha)}[\frac{\partial\bar{u}}{\partial y} + \frac{Aqu}{(1+Aqy)}]\}\ \Lambda$$

$$- \frac{i\sigma\Omega}{\beta\bar{\Theta}_c}\sin\alpha Ra_L^{-\omega^2}r \tag{2.3-10}$$

$$-Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+\bar{\gamma}_d)\bar{v}\frac{d^2\delta_1}{dy^2} + \{iB(1+\bar{\gamma}_d) - Ra_L^{-\frac{1}{5}(1+\omega^2)}[(1+\bar{\gamma}_d)\frac{\partial\bar{v}}{\partial y} + \frac{(1+\bar{\gamma}_d)}{(1+Aqy)}(2Aq\bar{v}$$

$$+ \frac{\partial\bar{u}}{\partial\bar{x}}) - \frac{\partial\bar{\gamma}_d\bar{v}}{\partial y}]\}\frac{d\delta_1}{dy} + \{iB[\frac{(1+\bar{\gamma}_d)Aq}{(1+Aqy)} + \frac{\partial\bar{\gamma}_d}{\partial y}] - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{2A}{(1+Aqy)^2}(q\bar{u}\frac{\partial\bar{\gamma}_d}{d\bar{x}}$$

$$+ (1+\bar{\gamma}_d)(\frac{\bar{u}dq}{d\bar{x}} + \frac{q\partial\bar{u}}{\partial\bar{x}})) + \frac{(1+\bar{\gamma}_d)}{(1+Aqy)}(\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{Aq\partial\bar{v}}{\partial y}) - \frac{2A^2qy\bar{u}(1+\bar{\gamma}_d)}{(1+Aqy)^3}\frac{dq}{d\bar{x}}$$

$$+ \frac{1}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})\frac{\partial\bar{\gamma}_d}{\partial y}]\}\delta_1 - (1+\bar{\gamma}_d)[\frac{Aq\bar{u}}{(1+Aqy)} + \frac{\partial\bar{u}}{\partial y}]\frac{d\delta_2}{dy}$$

$$- \{(1+\bar{\gamma}_d)[\frac{2Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} + \frac{\partial^2\bar{u}}{\partial y^2}] + \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}] + \frac{iBRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}\frac{\partial\bar{\gamma}_d}{\partial\bar{x}}\}\delta_2$$

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\frac{dr}{dy} - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{v}\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aq\bar{u}\partial\bar{v}}{\partial y} + \frac{2Aq\bar{v}\partial\bar{u}}{\partial y})$$

$$+ \frac{A\bar{u}}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}} + \frac{2q\partial\bar{u}}{\partial\bar{x}}) - \frac{A^2qy\bar{u}^2}{(1+Aqy)^3}\frac{dq}{d\bar{x}}]\ r$$

$$= -\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\bar{M}d^3\delta_1}{dy^3} - \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{2\partial\bar{M}}{\partial y} + \frac{2\bar{M}Aq}{(1+Aqy)} \pm \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}]\frac{d^2\delta_1}{dy^2}$$

$$- \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{\partial^2\bar{M}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{M}}{\partial y} + \frac{1}{(j\pm Jy\sin\alpha)}[\pm\frac{J\sin\alpha\partial\bar{M}}{\partial y}$$

$$\pm \frac{2\bar{M}AqJ\sin\alpha}{(1+Aqy)} - \frac{\bar{M}}{(j\pm Jy\sin\alpha)}(J^2\sin^2\alpha + \Omega^2)] - \frac{\bar{M}A^2q^2}{(1+Aqy)^2}\}\frac{d\delta_1}{dy}$$

$$+ \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{Aq}{(1+Aqy)}\frac{\partial^2\bar{M}}{\partial y^2} + \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial\bar{M}}{\partial y} - \frac{\bar{M}A^3q^3}{(1+Aqy)^3}$$

$$\mp \frac{AqJ\sin\alpha}{(1+Aqy)(j\pm Jy\sin\alpha)}\frac{\partial\bar{M}}{\partial y} + \frac{1}{(j\pm Jy\sin\alpha)^2}[\Omega^2\frac{\partial\bar{M}}{\partial y} + \frac{\bar{M}Aq}{(1+Aqy)}(J^2\sin^2\alpha$$

$$+\Omega^2)] \mp \frac{\bar{M}\Omega^2 J\sin\alpha}{(j\pm Jy\sin\alpha)^3}\}\delta_1 + \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{Aq\bar{u}}{(1+Aqy)} - \frac{\partial\bar{u}}{\partial y}]\frac{d^2\Lambda}{dy^2}$$

$$- \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{2\partial^2\bar{u}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q\bar{u}}{(1+Aqy)^2} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial\bar{u}}{\partial y}$$

$$+ \frac{Aq\bar{u}}{(1+Aqy)}]\}\frac{d\Lambda}{dy} - \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{\partial^3\bar{u}}{\partial y^3} + \frac{2Aq}{(1+Aqy)}\frac{\partial^2\bar{u}}{\partial y^2} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial\bar{u}}{\partial y}$$

$$+ \frac{A^3q^3\bar{u}}{(1+Aqy)^3} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2\bar{u}}{\partial y^2} + \frac{2Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y}] - \frac{J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}[\frac{\partial\bar{u}}{\partial y}$$

$$+ \frac{Aq\bar{u}}{(1+Aqy)}]\}\Lambda + \frac{\sigma Ra_L^{-\omega^2}}{\beta\bar{\Theta}_c}\frac{dr\sin\alpha}{dy} + \frac{\sigma Ra_L^{-\omega^2}\sin\alpha}{\beta\bar{\Theta}_c(1+Aqy)}[Aq + Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{d\alpha}{d\bar{x}}]r$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+\bar{\gamma}_d)\bar{c}(1+a)\bar{v}\frac{ds}{dy} + (1+\bar{\gamma}_d)[\frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{c}\bar{u}}{(1+Aqy)}\frac{da}{d\bar{x}} - iB\bar{c}(1+a)]s$$

$$+ (1+\bar{\gamma}_d)\bar{c}(1+a)\frac{\partial\bar{\theta}}{\partial y}\delta_2 + Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{(1+\bar{\gamma}_d)\bar{c}}{(1+Aqy)}[\bar{\theta}\frac{da}{d\bar{x}} + (1+a)\frac{\partial\bar{\theta}}{\partial\bar{x}}]\delta_1$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{(1+a)\bar{u}}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}} + \bar{v}(1+a)\frac{\partial\bar{\theta}}{\partial y}][\bar{c}r + (1+\bar{\gamma}_d)\Delta]$$

$$= (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{K}\frac{d^2s}{dy^2} + (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\bar{K}Aq}{(1+Aqy)} \pm \frac{\bar{K}J\sin\alpha}{(j\pm Jy\sin\alpha)}$$

$$+ \frac{\partial\bar{K}}{\partial y}]\frac{ds}{dy} - \frac{(1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\Omega^2\bar{K}s}{(j\pm Jy\sin\alpha)} + (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\partial\bar{\theta}}{\partial y}\frac{dT}{dy}$$

$$+ (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\partial^2\bar{\theta}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\bar{\theta}}{\partial y}]\,T$$

Equations 2.3-10 can be utilized in the stability analysis of any free-convection flow which satisfies the boundary-layer equations 2.1-3 and the boundary conditions 2.1-4, assuming that the disturbance takes the form of a set of roll vortices which satisfies the assumptions involved in obtaining equations 2.3-10. However, a set of auxiliary relationships for r, Λ, Δ and T and a set of boundary conditions are required before attempting a solution.

As stated in section 2.3-3, the present study is primarily concerned with a study of constant-property, free-convection flows over two-dimensional surfaces. Using the Boussinesq approximation for the density as in section 2.3-3 and assuming that $\varepsilon \ll 1$, the disturbance equations reduce to

$$\frac{d\delta_2}{dy} + \frac{Aq}{(1+Aqy)}\delta_2 + i\Omega\delta_3 = 0.$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{d^2\delta_3}{dy^2} + [Ra_L^{-\frac{1}{5}(1+\omega^2)}\frac{\partial\bar{v}}{\partial y} - iB]\frac{d\delta_3}{dy} - iRa_L^{-\frac{1}{5}(1+\omega^2)}\Omega\bar{v}\frac{d\delta_2}{dy}$$

$$- [B + iRa_L^{-\frac{1}{5}(1+\omega^2)}\frac{\partial\bar{v}}{\partial y}]\delta_2 + \frac{2i\Omega Aq\bar{u}}{(1+Aqy)}\delta_1$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{d^3\delta_3}{dy^3} + \frac{Aq}{(1+Aqy)}\frac{d^2\delta_3}{dy^2} - [\frac{A^2q^2}{(1+Aqy)^2} + \Omega^2]\frac{d\delta_3}{dy} - \frac{i\Omega d^2\delta_2}{dy^2}$$

$$- \frac{i\Omega Aq}{(1+Aqy)}\frac{d\delta_2}{dy} + [\frac{i\Omega A^2q^2}{(1+Aqy)^2} + i\Omega^3]\delta_2\} - i\Omega\sigma Ra_L^{-\omega^2} s\cos\alpha$$

$$iRa_L^{-\frac{1}{5}(1+\omega^2)}\Omega\bar{v}\frac{d\delta_1}{dy} + \Omega[B + \frac{iRa_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\delta_1 + i\Omega[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\delta_2$$

$$= i\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{\Omega d^2\delta_1}{dy^2} + \frac{\Omega Aq}{(1+Aqy)}\frac{d\delta_1}{dy} - [\frac{\Omega A^2q^2}{(1+Aqy)^2} + \Omega^3]\delta_1\}$$

$$+ i\sigma\Omega Ra_L^{-\omega^2} s\sin\alpha$$

(2.3-11)

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{d^2\delta_1}{dy^2} + \{iB - Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{\partial\bar{v}}{\partial y} + \frac{1}{(1+Aqy)}(2Aq\bar{v} + \frac{\partial\bar{u}}{\partial\bar{x}})]\}\frac{d\delta_1}{dy} + \{\frac{iBAq}{(1+Aqy)}$$

$$- Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{2A}{(1+Aqy)^2}(\bar{u}\frac{dq}{d\bar{x}} + q\frac{\partial\bar{u}}{\partial\bar{x}}) + \frac{1}{(1+Aqy)}(\frac{\partial^2\bar{u}}{\partial y\partial\bar{x}} + Aq\frac{\partial\bar{v}}{\partial y})$$

$$- \frac{2A^2qy\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}}]\}\delta_1 - [\frac{Aq\bar{u}}{(1+Aqy)} + \frac{\partial\bar{u}}{\partial y}]\frac{d\delta_2}{dy} - [\frac{2Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} + \frac{\partial^2\bar{u}}{\partial y^2}]\delta_2$$

$$= -\sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{d^3\delta_1}{dy^3} + \frac{2Aq}{(1+Aqy)}\frac{d^2\delta_1}{dy^2} - [\Omega^2 + \frac{A^2q^2}{(1+Aqy)^2}]\frac{d\delta_1}{dy}$$

$$- \frac{Aq}{(1+Aqy)}[\Omega^2 - \frac{A^2q^2}{(1+Aqy)^2}]\delta_1\} - \sigma Ra_L^{-\omega^2}\frac{ds}{dy}\sin\alpha - \frac{\sigma Ra_L^{-\omega^2}\sin\alpha}{(1+Aqy)}[Aq$$

$$+ Ra_L^{-\frac{1}{5}(1+\omega^2)} \frac{d\alpha}{d\bar{x}}]s$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+a)\bar{v}\frac{ds}{dy} + [\frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{u}}{(1+Aqy)}\frac{da}{d\bar{x}} - iB(1+a)]s + (1+a)\frac{\partial\bar{\theta}}{\partial y}\delta_2$$

$$+ \frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}[\bar{\theta}\frac{da}{d\bar{x}} + (1+a)\frac{\partial\bar{\theta}}{\partial\bar{x}}]\delta_1 = (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{d^2s}{dy^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{ds}{dy} - \Omega^2 s]$$

The above set of equations appears to be over-determined since there are five equations in four unknowns. However, a closer inspection reveals that the fourth equation can be obtained from the second and third equations by the procedure which follows. Firstly, multiply the third equation by (1+Aqy) and then take the derivative with respect to y of the resulting equation. Secondly, take the derivative with respect to x* of the second equation, and note that some of the terms are of the order of the terms previously neglected and, therefore, these terms must also be neglected. Finally, add the two modified equations and multiply the result by $\frac{i}{\Omega(1+Aqy)}$.

The first of equations 2.3-11 can be used to eliminate δ_3 from the other equations in the set. The resulting set of independent equations is the following:

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{d^3\delta_2}{dy^3} + \{Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{Aq\bar{v}}{(1+Aqy)} + \frac{\partial\bar{v}}{\partial y}] - iB\}\frac{d^2\delta_2}{dy^2}$$

$$+ \{Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{Aq}{(1+Aqy)}\frac{\partial\bar{v}}{\partial y} - \frac{2A^2q^2\bar{v}}{(1+Aqy)^2} - \Omega^2\bar{v}] - \frac{iBAq}{(1+Aqy)}\}\frac{d\delta_2}{dy}$$

$$+ \{Ra_L^{-\frac{1}{5}(1+\omega^2)}[\frac{4A^3q^3\bar{v}}{(1+Aqy)^3} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial\bar{v}}{\partial y} - \frac{\Omega\partial\bar{v}}{\partial y}] + iB[\frac{A^2q^2}{(1+Aqy)^2}$$

$$+ \Omega]\}\delta_2 + \frac{2\Omega^2Aq\bar{u}}{(1+Aqy)}\delta_1$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{d^4\delta_2}{dy^4} + \frac{2Aq}{(1+Aqy)}\frac{d^3\delta_2}{dy^3} - [\frac{3A^2q^2}{(1+Aqy)^2} + 2\Omega^2]\frac{d^2\delta_2}{dy^2}$$

$$+ \frac{Aq}{(1+Aqy)}[\frac{5A^2q^2}{(1+Aqy)^2} - 2\Omega^2]\frac{d\delta_2}{dy} + [\Omega^4 + \frac{2\Omega^2A^2q^2}{(1+Aqy)^2} - \frac{7A^4q^4}{(1+Aqy)^4}]\delta_2\}$$

$$- \Omega^2\sigma Ra_L^{-\omega^2}s\cos\alpha \tag{2.3-12}$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{v}\frac{d\delta_1}{dy} + [\frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v}) - iB]\delta_1 + [\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\delta_2$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\{\frac{d^2\delta_1}{dy^2} + \frac{Aq}{(1+Aqy)}\frac{d\delta_1}{dy} - [\frac{A^2q^2}{(1+Aqy)^2} + \Omega^2]\delta_1\} + \sigma Ra_L^{-\omega^2}s\sin\alpha$$

$$Ra_L^{-\frac{1}{5}(1+\omega^2)}(1+a)\bar{v}\frac{ds}{dy} + [\frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{u}}{(1+Aqy)}\frac{da}{d\bar{x}} - iB(1+a)]s + (1+a)\frac{\partial\bar{\theta}}{\partial y}\delta_2$$

$$+ \frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}}{(1+Aqy)}[\bar{\theta}\frac{da}{d\bar{x}} + (1+a)\frac{\partial\bar{\theta}}{\partial\bar{x}}]\delta_1$$

$$= (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{d^2s}{dy^2} + \frac{Aq}{(1+Aqy)}\frac{ds}{dy} - \Omega^2 s\right]$$

Following the procedure of section 2.3-3, the mean flow is assumed to be parallel. This assumption reduces equations 2.3-12 to the following set of equations:

$$- iB\frac{d^2\delta_2}{dy^2} - \frac{iBAq}{(1+Aqy)}\frac{d\delta_2}{dy} + iB\left[\frac{A^2q^2}{(1+Aqy)^2} + \Omega\right]\delta_2 + \frac{2\Omega^2 Aq\bar{u}}{(1+Aqy)}\delta_1$$

$$= \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left\{\frac{d^4\delta_2}{dy^4} + \frac{2Aq}{(1+Aqy)}\frac{d^3\delta_2}{dy^3} - \left[\frac{3A^2q^2}{(1+Aqy)^2} + 2\Omega^2\right]\frac{d^2\delta_2}{dy^2}\right.$$

$$\left. + \frac{Aq}{(1+Aqy)}\left[\frac{5A^2q^2}{(1+Aqy)^2} - 2\Omega^2\right]\frac{d\delta_2}{dy} + \left[\Omega^4 + \frac{2\Omega^2A^2q^2}{(1+Aqy)^2} - \frac{7A^4q^4}{(1+Aqy)^4}\right]\delta_2\right\}$$

$$- \Omega^2\sigma Ra_L^{-\omega^2} s\cos\alpha \tag{2.3-13}$$

$$- iB\delta_1 + \left[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}\right]\delta_2 = \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)}\left\{\frac{d^2\delta_1}{dy^2} + \frac{Aq}{(1+Aqy)}\frac{d\delta_1}{dy} - \left[\frac{A^2q^2}{(1+Aqy)^2}\right.\right.$$

$$\left.\left. + \Omega^2\right]\delta_1\right\} + \sigma Ra_L^{-\omega^2} s\sin\alpha$$

$$\left[\frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{u}}{(1+Aqy)}\frac{da}{d\bar{x}} - iB(1+a)\right]s + (1+a)\frac{\partial\bar{\theta}}{\partial y}\delta_2 + \frac{Ra_L^{-\frac{1}{5}(1+\omega^2)}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}}\delta_1$$

$$= (1+a)Ra_L^{-\frac{1}{5}(1+\omega^2)}\left[\frac{d^2s}{dy^2} + \frac{Aq}{(1+Aqy)}\frac{ds}{dy} - \Omega^2 s\right]$$

The appropriate boundary conditions to be satisfied by equations 2.3-13 follow from the conditions that the disturbance velocities vanish along the surface y = 0 and far from the surface. In addition, the disturbance temperature must vanish far from the surface and satisfy the energy balance at the surface as given by the expression 2.3-8. These boundary conditions may be written as:

$$y = 0 : \delta_1 = \delta_2 = \frac{d\delta_2}{dy} = 0$$

and

$$s = \frac{iL}{Bb} Ra_L^{-\frac{2}{5}(1+\omega^2)}\frac{ds}{dy} \qquad (2.3\text{-}14)$$

$$y \to \infty : \delta_1 = \delta_2 = \frac{d\delta_2}{dy} = s = 0.$$

CHAPTER III

SOLUTION OF THE PROBLEM

In chapter 2, the formulation of the problem is presented in two steps: the formulation of the steady-flow or mean-flow equations and the formulation of the disturbance equations. The resulting sets of equations from these steps are distinct in character and, therefore, they require different methods of solution. This chapter presents the methods of solution adopted for the present analysis.

3.1 Solution of the Mean-Flow Equations

There are several possible approaches to solving the mean-flow equations. For example, given the property variations, an explicit finite-difference scheme might be applied to equations 2.1-3 subject to the boundary conditions 2.1-4. An approach such as this was attempted in the present investigation but it was abandoned due to difficulties in starting the solution at $x = 0$ because of the singularity at $x = 0$ and because of the problem of obtaining accurate approximations to the derivatives with respect to x in this region. In addition, explicit finite-difference schemes have an inherent instability.

A second possible approach would be to transform the set of equations 2.1-3 by a transformation such as the von Mises transformation. In the transformed plane, the boundary-layer thickness is more uniform throughout the region of interest. In addition to the transformation, the explicit finite-difference scheme may be replaced by an implicit finite-difference scheme, thereby avoiding any inherent instability. However, the von Mises transformation introduces a singularity at the

surface, and this singularity may be difficult to handle in the solution. This approach was also tried, but it was abandoned because of difficulties attributed to the singularity.

A third finite-difference scheme, which is similar to the second scheme, is given in a paper by Patankar and Spalding [47]. Patankar and Spalding suggested a transformation of the equations of the von Mises type, but which uses a new definition of the nondimensional stream function. The authors stated that the advantage of their method was that it provided a definite bound on the boundary-layer thickness. However, their method still has the disadvantage of introducing a singularity at the surface.

Integral methods offer a fourth possible approach to the solution of the mean-flow equations. Levy [23] investigated the possibility of using integral methods in free-convection problems. Michiyoshi [24] also used an integral method in his analysis. The disadvantage of integral methods is their inability to accurately predict the shape and magnitude of the velocity and temperature profiles. Since the influence of these profiles on the stability analysis is not yet clear, it is felt that an attempt should be made to keep the errors in the profiles to a minimum.

Another possible solution approach is to use a perturbation expansion. Kierkus [25] used a perturbation expansion in terms of a small Grashof-number parameter to study the effect of inclination from the vertical on laminar, free-convection flow about an isothermal plate. Although the results of this perturbation analysis are in good agreement with experimental results, this approach is limited by the size of the perturbation parameter.

The difficulties which may be encountered in utilizing any of the above methods suggest that perhaps some other approach might be sought. For the transformed boundary-layer equations 2.2-9 subject to the boundary-conditions 2.2-10, it is possible to use another approach; however, the method is not restricted to the flows described by these equations, but instead it can be applied to any set of equations expressed in terms of a Falkner-Skan transformation. It is noted in chapter 2 and appendix B how the Falkner-Skan transformation might be extended to more general flows.

Basically, the scheme chosen replaces the schemes mentioned above by a numerical integration, for which there are two broad choices: the equations may be integrated as a single set of partial differential equations, or the dependent variables can be expanded using, for example, a Goertler expansion which leads to an infinite set of ordinary differential equations. An important consideration in applying the latter method is the number of terms needed in the expansion to obtain the desired accuracy in the solution. Although it is believed that either of the above methods is suitable, the former procedure is chosen for the present analysis. The latter method was used by Kuiken [16] in a study of axisymmetric free-convection boundary-layer flows past vertical cylinders and cones. Saville and Churchill [48] also used a Goertler expansion in their analysis of horizontal cylinders and vertical axisymmetric bodies.

The numerical integration in the former method is made possible by replacing the partial derivatives with respect to ξ by finite-difference approximations as discussed by Hartree and Womersley [49]. The method has been applied to laminar, forced-convection boundary-layer flows for some time, but the method has only recently been adopted to free-convection boundary-layer problems. After applying a Falkner-Skan transformation,

Hayday, Bowlus and McGraw [18] applied this method of solution to a free-convection flow past a vertical plate with a step discontinuity in the surface temperature. The apparent success of the method in handling such a difficult boundary condition suggested that this approach might also be applied to the present work.

After replacing the partial derivatives with respect to ξ by finite-difference approximations, the equations reduce to ordinary differential equations for any value of ξ. In order to solve the equations for a particular ξ, it is assumed that the equations have been solved at all previous steps in the ξ-direction. At $\xi = 0$, it is assumed that the derivatives with respect to ξ remain finite, and therefore, these derivatives are eliminated because of the coefficient ξ. However, ξ cannot be set to zero uniformly throughout the equations. This is a consequence of the coefficient of the pressure term in the lateral momentum equation. In order to start the procedure, ξ must be given a small non-zero value in this term.

For the finite-difference approximations to the ξ-derivatives, a two-point difference formula must be used for the first step beyond $\xi = 0$. For subsequent steps, a three-point formula may be used. The two- and three-point formulae were given by Smith and Clutter [50] as well as by Hayday, Bowlus and McGraw [8], but they are repeated here for completeness. For any variable, say S, the two- and three-point formulae at $\xi = \xi_j$ are:

$$\left(\frac{\partial S}{\partial \xi}\right)\Big|_{\xi=\xi_j} \simeq \frac{S_j - S_{j-1}}{\xi_j - \xi_{j-1}}$$

and

$$\left(\frac{\partial S}{\partial \xi}\right)\Big|_{\xi=\xi_j} \simeq \left[\frac{1}{(\xi_j-\xi_{j-1})} + \frac{1}{(\xi_j-\xi_{j-2})}\right] S_j - \left[\frac{(\xi_j-\xi_{j-2})}{(\xi_j-\xi_{j-1})(\xi_{j-1}-\xi_{j-2})}\right] S_{j-1}$$

$$+ \left[\frac{(\xi_j-\xi_{j-1})}{(\xi_j-\xi_{j-2})(\xi_{j-1}-\xi_{j-2})}\right] S_{j-2}$$

respectively, where S represents f, f_ζ, π, χ or a. It should be noted that a four-point formula may be used beyond the third step. Smith and Clutter experimented with two-, three- and four-point formulae and found that the three- and four-point formulae were stable and accurate. In fact, they found that the three-point formula was sufficiently accurate such that the four-point formula was not required. Since Smith and Clutter were concerned with forced-convection flows, it does not necessarily follow that the results of these numerical experiments are valid for free-convection flows. However, only the two- and three-point formulae are used in the present work.

After suitably discretizing the ξ-derivatives in the equations, the resulting equations may be solved at each ξ by any of the methods used in obtaining similarity solutions. For the present work, the equations are forward-integrated, starting at $\zeta = 0$, using a fourth-order Runge-Kutta procedure. To start the integration, it is necessary to guess values for $f_{\zeta\zeta}(0, \xi)$, $\chi_\zeta(0, \xi)$ and $\pi(0, \xi)$, which are the eigenvalues. An iterative process similar to that followed by Nachtsheim and Swigert [51] is used to determine the eigenvalues such that the asymptotic boundary conditions for $\zeta \to \infty$ are satisfied within a specified accuracy, e.g. 10^{-6}. After making the initial guesses for the eigenvalues, successive approximations are obtained by means of a modified Newton-Raphson procedure which incorporates the asymptotic nature of the boundary layer. That is, to the conditions

$$f_\zeta(\zeta_\infty, \xi) = \chi(\zeta_\infty, \xi) = \pi(\zeta_\infty, \xi) = 0,$$

the following conditions are added:

$$f_{\zeta\zeta}(\zeta_\infty, \xi) = \chi_\zeta(\zeta_\infty, \xi) = \pi_\zeta(\zeta_\infty, \xi) = 0,$$

where ζ_∞ denotes the boundary-layer "edge". A detailed discussion of this

procedure was given by Nachtsheim and Swigert, but a brief description is also given here.

After introducing the following definitions

$\Gamma_{1n} = f_{\zeta\zeta}\ (0,\ \xi_n)$,

$\Gamma_{2n} = \chi_\zeta\ (0,\ \xi_n)$

and $\Gamma_{3n} = \pi\ (0,\ \xi_n)$,

where ξ_n is the nth step in the ξ-direction, the first step of the iterative process consists of determining

$$E_n = \sum_{i=1}^{6} [Q_n^{(i)} + \frac{\partial Q_n^{(i)}}{\partial \Gamma_{1n}} \Delta\Gamma_{1n} + \frac{\partial Q_n^{(i)}}{\partial \Gamma_{2n}} \Delta\Gamma_{2n} + \frac{\partial Q_n^{(i)}}{\partial \Gamma_{3n}} \Delta\Gamma_{3n}]^2,$$

where $Q_n^{(1)} = f_\zeta\ (\zeta_\infty,\ \xi_n)$,

$Q_n^{(2)} = f_{\zeta\zeta}\ (\zeta_\infty,\ \xi_n)$,

$Q_n^{(3)} = \chi\ (\zeta_\infty,\ \xi_n)$,

$Q_n^{(4)} = \chi_\zeta\ (\zeta_\infty,\ \xi_n)$,

$Q_n^{(5)} = \pi\ (\zeta_\infty,\ \xi_n)$

and $Q_n^{(6)} = \pi_\zeta\ (\zeta_\infty,\ \xi_n)$.

Next, this quantity is <u>minimized</u> with respect to the increments $\Delta\Gamma_{1n}$, $\Delta\Gamma_{2n}$ and $\Delta\Gamma_{3n}$. The incremental changes in Γ_{1n}, Γ_{2n} and Γ_{3n} necessary to effect this minimization are then found and used to modify the eigenvalues.

It must be noted that the above process will converge only if the initial approximations to the eigenvalues lie within certain limits of the correct eigenvalues, these limits being the boundaries of the region of convergence. To facilitate the solution for the eigenvalues, the error indicated by E_{nj} is minimized at several $\zeta_j < \zeta_\infty$ in order to keep the set of approximate eigenvalues within a region of convergence. Since the value of ζ_∞ is unknown <u>a priori</u>, this procedure may also be used to

find a suitable measure of ζ_∞. The procedure assumes that for any reasonable set of approximate eigenvalues there will be some $\zeta_1 < \zeta_\infty$ for which E_{n1} lies within a fairly large region of convergence. It seems reasonable to expect that as ζ_j becomes larger, the region of convergence becomes progressively smaller. Having minimized E_{n1} and modified the approximate eigenvalues, the equations are forward-integrated from zero to $\zeta_2 > \zeta_1$. This procedure is repeated m times until E_{nm} is less than a specified value, e.g. 10^{-6}. The corresponding value of ζ_m then defines ζ_∞. Clearly this method permits a much wider range of initial approximations to the eigenvalues for which the process will converge; however, the choice of ζ_1 and the optimum number of steps m can be determined only by trial and error methods.

The above procedure for finding ζ_∞ was used initially; but after finding a suitable value, ζ_∞ was specified to provide a constant value for all ξ such that for any particular ξ the necessary information at previous values of ξ would be available for all $\zeta \leq \zeta_\infty$. The value specified varied with the surface inclination and corresponded to a value for y_∞ at $\xi = 1.0$. At $\alpha = 0$, y_∞ at $\xi = 1.0$ was given a value of 10.0, while at $\alpha = \pi/2$, it was given a value of 8.0. For other angles, y_∞ at $\xi = 1.0$ varied between these limits.

The utilization of the Falkner-Skan formulation in conjunction with the local similarity concept has several advantages over other possible methods. It provides a definite procedure for starting the solution at $\xi = 0$, and the boundary-layer thickness remains finite at $\xi = 0$ since the procedure removes some of the ξ-variation of the transformed boundary-layer thickness. In addition to these advantages, the most important advantage is that, for a given ξ, the equations reduce to ordinary

differential equations.

3.2 Solution of the Disturbance Equations

3.2-1 Tollmien-Schlichting Wave Disturbances

This section presents a discussion of some of the numerical procedures used in previous work as well as the procedure used in the present work. This discussion refers specifically to equations 2.3-6 and the boundary conditions 2.3-7 and 2.3-8, but the procedures can also be applied without modification to equations 2.3-5. The basic ideas of these procedures could also be applied in a solution attempt on equations 2.3-3, however, the procedures would have to be modified to handle the additional relationships for r, Λ, Δ and T.

Essentially, each of the procedures involves two solutions, an inner solution and an outer solution, and these must be matched. The outer solution satisfies the simplified disturbance equations, which are obtained by neglecting the terms involving mean-flow quantities since the mean-flow quantities are assumed to be approaching zero, and is valid for large values for the independent variable. The inner solution is obtained by integrating the disturbance equations over a range of the independent variable which includes most of the mean-flow boundary layer.

The set of disturbance equations corresponding to the inner solution constitutes a two-point boundary-value problem for which three complex conditions are specified at each of the boundaries. Therefore, to start the integration at one end and proceed to the other end, three unknown complex conditions at the starting point must be guessed. Since the disturbance momentum equation is a fourth-order, linear, homogeneous differential equation, the second derivative of the disturbance stream function amplitude evaluated at the starting point determines the scale

of the disturbances and may be assigned arbitrarily. In addition to the remaining two unknown conditions, one complex parameter or two real parameters must be determined. Thus, in total, there are six real or three complex eigenvalues to be determined.

Nachtsheim [33] chose the complex wave velocity as the unknown parameter, and since he integrated from the surface to a suitably chosen edge, the other two complex eigenvalues were $\phi'''(0)$ and $S'(0)$. For a given wavenumber-Reynolds number pair, Nachtsheim obtained successive approximations to the eigenvalues by applying a Newton-Raphson procedure based on the conditions at the outer point of his integration range. To find a point on the neutral stability curve, he had to fix the Reynolds number and vary the wavenumber until the imaginary part of the complex wave velocity just changed sign. He then repeated this procedure at other Reynolds numbers to obtain the neutral stability curve.

Knowles and Gebhart [36] followed a procedure somewhat like that used by Nachtsheim except they chose the real parts of the wavenumber and of the frequency as the unknown parameters. They set the imaginary part of the wavenumber to zero, as did Nachtsheim, and they also set the imaginary part of the frequency to zero. Therefore, for a given Grashof number, a point was obtained on the neutral stability curve. This procedure requires much less computation to obtain the neutral stability curve than Nachtsheim's method. Another modification introduced by Knowles and Gebhart was the boundary condition for the disturbance temperature difference as discussed in section 2.3-3.

Dring and Gebhart [37] encountered a problem with the procedures used by Nachtsheim and by Knowles and Gebhart, namely that the solutions would become unbounded if the integration proceeded far enough from the

surface. This effect is independent of the accuracy of the eigenvalues and is related to the exponential nature of the solutions in the region far from the surface. Theoretically the coefficients of the terms with positive real parts in their exponents must vanish because these solutions are physically inadmissible. However, numerically these coefficients are very small but still non-zero. Therefore, the terms with positive real parts in their exponents will eventually dominate the solution if the integration proceeds far enough. To avoid this problem, Dring and Gebhart reversed the direction of integration. They used the outer solution to start the integration by choosing the coefficients of the terms with positive real parts in their exponents to be identically zero. Of the remaining three complex coefficients, one coefficient was used to determine the scale of the solutions and the remaining two became two of the complex eigenvalues. In addition, they also chose the real parts of the wavenumber and of the frequency as eigenvalues. Successive approximations to the eigenvalues were obtained by a Newton-Raphson procedure applied to the conditions at the surface.

Although each of the above procedures appears to be relatively simple to apply, in practice this is not the case. Without having at least one complete set of eigenvalues corresponding to one point on the neutral stability curve, it is very difficult to guess six real or three complex numbers which will be close enough to the correct values for the iterative process to converge. This difficulty can be appreciated by realizing that the search technique amounts to searching for a point in a six-dimensional space. This problem can be overcome by noting that the disturbance equations can be combined to form a sixth-order, linear, homogeneous differential equation which can be solved by a linear combination

of six linearly independent solutions. Hieber and Gebhart [39] used this approach while following Dring and Gebhart's procedure by integrating from the outer edge to the surface; but instead of guessing the coefficients as Dring and Gebhart did, Hieber and Gebhart integrated the solutions separately. They used one of the three non-zero coefficients of the linear combination to scale the solution and solved for the other two coefficients by satisfying two of the boundary conditions at the surface. Hieber and Gebhart used the complex wavenumber as the remaining eigenvalue. The wavenumber was varied until the remaining boundary condition was satisfied within the specified limit. This approach simplified the solution to a search for one complex eigenvalue rather than for three complex eigenvalues.

The procedure followed by Hieber and Gebhart appears to be far superior to the procedures used previously, and therefore, this procedure is used in the present work. It is assumed that the disturbance amplitudes can be expressed as:

$$\phi = \phi_1 + C_2\phi_2 + C_3\phi_3$$

and $$s = s_1 + C_2 s_2 + C_3 s_3$$

At the starting point of the integration, the outer solutions indicate that

$$\phi_1(y_e) = e^{-\lambda y_e},$$

$$s_1(y_e) = 0,$$

$$\phi_2(y_e) = e^{-Cy_e},$$

$$s_2(y_e) = 0,$$

$$\phi_3(y_e) = e^{-Dy_e}$$

and $$s_3(y_e) = \frac{(D^2 - \lambda^2)(D^2 - C^2)}{(i\lambda\cos\alpha - D\sin\alpha)} \; Ra_L^{-\frac{1}{5}(1-\omega^2)} \, e^{-Dy_e}$$

where y_e is the value of y at the starting point of the integration and

$$C = [\lambda^2 - \frac{iB}{\sigma} Ra_L^{\frac{1}{5}}(1+\omega^2)]^{1/2}$$

and $$D = [\lambda^2 - iB\, Ra_L^{\frac{1}{5}}(1+\omega^2)]^{1/2}$$

The forms of the outer solutions given in the above expression are obtained in exactly the same way as those obtained by Nachtsheim and by Dring and Gebhart.

In their work, Hieber and Gebhart used the wavenumber as the complex eigenvalue, but the present work uses the real parts of the wavenumber and of the frequency as unknown real eigenvalues. After making initial guesses for λ_r and B_r and separately integrating the three solutions from $y = y_e$ to $y = 0$, the boundary conditions

$$\phi(0) = 0$$

and $$\phi'(0) = 0$$

are used to solve for C_2 and C_3. Then the condition on $s(0)$ is used to test the approximate values for λ_r and B_r. If this condition is not satisfied within the specified accuracy, e.g. 10^{-5}, then a Newton-Raphson procedure is applied to the condition. For the present work $s(0) = 0$ is used as the boundary condition, and therefore, the Newton-Raphson procedure is of the form:

$$0 \simeq s_r(0) + \frac{\partial s_r(0)}{\partial \lambda_r} \Delta\lambda_r + \frac{\partial s_r(0)}{\partial B_r} \Delta B_r$$

and $$0 \simeq s_i(0) + \frac{\partial s_i(0)}{\partial \lambda_r} \Delta\lambda_r + \frac{\partial s_i(0)}{\partial B_r} \Delta B_r.$$

The partial derivatives in this procedure are approximated by finite differences, and λ_r and B_r are changed to 1.001 λ_r and 1.001 B_r to find these derivatives. With the new values of λ_r and B_r determined from this process, the integrations are repeated and the iteration continues until $s(0) = 0$ is satisfied within the prescribed limits.

3.2-2 Taylor-Goertler Roll-Vortex Disturbances

The methods of solution of the disturbance equations for the wave disturbances can also be applied to the equations for the roll-vortex disturbances. However, there are some important differences in the equations which result in considerable simplification. Firstly, the real part of B is zero since the vortices may be considered as standing transverse waves. Secondly, if the vortices are assumed to be neutrally stable with position and time, the imaginary parts of Ω and B are also zero. Under these simplifications the disturbance equations reduce to real equations.

Equations 2.3-11, 2.3-12 and 2.3-13 represent sets of linear, homogeneous ordinary differential equations of eighth order. In order to solve these equations, a combination of eight linearly independent solutions is required; but following the procedure in the previous section, four of the coefficients must be zero because the corresponding outer solutions have positive real parts in their exponents. Another coefficient is set to unity to scale the solutions and the remaining three coefficients are solved for by satisfying three of the four boundary conditions given in 2.3-14. If the above simplifications are used, the equations are real and the remaining boundary condition is used in finding the real part of the wavenumber in the same way as λ_r and B_r are obtained for the wave disturbance.

For the present work, the surface is isothermal and, therefore, the function $a(\bar{x})$ is zero. This makes it possible to reduce equations 2.3-13 to a sixth-order set, thus simplifying the procedure even further. Under these conditions, the disturbance amplitude functions are expressed as

$$\delta_2 = \delta_{21} + C_2\delta_{22} + C_3\delta_{23}$$

and $$s = s_1 + C_2 s_2 + C_3 s_3$$

At the starting point of the integration, y_e, the solutions take the form of the outer solutions as follows:

$$\delta_{21}(y_e) = e^{-\Omega_r y_e},$$

$$s_1(y_e) = 0,$$

$$\delta_{22}(y_e) = y_e\, e^{-\Omega_r y_e},$$

$$s_2(y_e) = 0,$$

$$\delta_{23}(y_e) = y_e{}^2\, e^{-\Omega_r y_e}$$

and $$s_3(y_e) = \frac{8}{\cos\alpha}\, Ra_L^{-\frac{1}{5}(1-\omega^2)}\, e^{-\Omega_r y_e}.$$

Since $e^{\pm\Omega_r y}$ are the only forms of the outer solutions for δ_{21}, it is necessary to use $ye^{\pm\Omega_r y}$ and $y^2 e^{\pm\Omega_r y}$ to form the other solutions. The remainder of the procedure closely parallels that described in section 3.2-1 and therefore will not be repeated.

CHAPTER IV

RESULTS AND DISCUSSION

4.1 Mean Flow

4.1-1 Errors

Solutions of the transformed boundary-layer equations 2.2-9 subject to the boundary conditions 2.2-10 have been obtained for air (Prandtl number of 0.72) adjacent to a flat, two-dimensional, isothermal surface. The solutions have been generated at positions along the surface up to X = L, where the Rayleigh number is 10^6. All of the numerical computations have been done in double precision on an IBM 360/67 computer at the University of Alberta. In the computations, the value of $\Delta\zeta$ was chosen such that $\Delta y = 0.05$ at $\xi = 1.0$ and this step size was used in all solutions. As a check on the truncation error in the calculations, this step size was doubled with the result that the eigenvalues changed in or beyond the third decimal place. A value of 10^{-4} was used for $\xi°$ in most of the calculations, but variations from 10^{-1} to 10^{-6} were also used with the result that the eigenvalues changed in the third decimal place at the most. For the finite-difference approximations of the partial derivatives with respect to ξ, the step size $\Delta\xi$ was decreased until the eigenvalues obtained from two step sizes agreed to at least three figures for values of $\xi \geq 0.1$. The resulting step size for the first step away from $\xi = 0$ was quite small to minimize the error in the two-point finite-difference approximation which is of the order of $\frac{\Delta\xi}{2}$ per step. As the solution proceeded in the ξ-direction, the value of $\Delta\xi$ had to be increased several times because of a numerical stability problem. In most cases, it was found that the ratio of $\frac{\xi}{\Delta\xi}$ had to be less than about 3. Because of this

restriction, the last step size used in many solutions was $\Delta\xi = 0.4$. This would appear to give a considerable error since the error per step using a three-point finite-difference formula is equal to $\frac{(\Delta\xi)^2}{3}$ multiplied by the third derivative of the functions with respect to ξ. However, the solutions reveal that the functions in question are almost asymptotic in the vicinity of $\xi = 1.0$, and therefore, the third derivatives with respect to ξ would be very small as would be the error in the finite-difference approximation. The results of these computations are presented in figures 2 to 7.

4.1-2 Mean-Flow Results

Figures 2 snd 3 present typical examples of the effect of surface inclination on the temperature, pressure and velocity profiles at different positions along the surface. Figure 2 presents the results for a position given by $\xi = 0.2$, whereas figure 3 considers the position $\xi = 1.0$. The temperature and velocity profiles are shown for 0°, 30°, 45° and 90° in figure 2 and for 0°, 30°, 45°, 60° and 90° in figure 3. The pressure profiles are shown for 0°, 30°, 45°, 60°, 75°, 105° and 120° in both figures. The eigenvalues for these profiles are tabulated in table 1.

Figures 4 and 5 indicate typical examples of the variations with position along the surface of the temperature, pressure and velocity profiles for specific inclinations. Figure 4 considers the profiles for 45° at the positions $\xi = 0.00625$, 0.01875, 0.04375, 0.1, 0.2, 0.4, 0.6 and 1.0. Figure 5 presents the profiles for 75° at the positions $\xi = 0.00625$, 0.025, 0.05, 0.1, 0.2, 0.4, 0.6, and 1.0. For these profiles, the eigenvalues are listed in table 2.

With the exceptions of the flows for 0° and 90°, the mean flow is non-similar and two examples of the departures from similarity of the

temperature, pressure and velocity are indicated in figures 6 and 7. Figure 6 shows this departure for a surface inclination of 45° at the positions ξ=0.00625, 0.01875, 0.04375, 0.1, 0.4, and 1.0. Figure 7 shows the departure for an inclination of 75° at the position ξ=0.00625, 0.01875, 0.1, 0.4 and 1.0. The eigenvalues for these profiles are also listed in table 2.

4.1-3 Discussion of the Results

Figures 2 and 3 clearly show that for any position along the surface the mean-flow boundary-layer thickness decreases and the heat transfer increases markedly as the surface inclination changes from 0° to 30°. The boundary-layer thickness continues to decrease and the heat transfer continues to increase at much slower rates as the inclination increases from 30° to 90° at which the boundary-layer thickness reaches a minimum and the heat transfer reaches a maximum. As is indicated by the order-of-magnitude analysis in appendix A, the changes in the profiles from 0° to 30° are due primarily to the decrease in the lateral buoyancy force and, therefore, to the decrease in the lateral pressure gradient as well. For inclinations from 30° to 90°, the changes in the profiles are due primarily to the increase in the longitudinal buoyancy force. This longitudinal buoyancy force varies from zero at0° to a maximum at 90°, whereas the lateral buoyancy force varies from a maximum at 0° to zero at 90°.

Figures 2b and 3b indicate the variation in the pressure profiles as the surface varies from 0° to 120°. The pressure changes sign as the inclination passes through 90°, but for inclinations on opposite sides of the vertical, the magnitudes of the pressure are almost identical. This fact by itself is evidence that the pressure has little effect on the

flow at least over the range of inclinations from 60° to 120°. In support of this conclusion, the velocity and temperature profiles for 75° and 105° as well as for 60° and 120° were found to be very nearly coincident. None of these profiles are included in figure 2 since the departures from the 90° profiles are very small. In fact for most purposes, the 90° profiles could be used for any angle from 60° to 120°. However, figure 3 does show the velocity and temperature profiles at 60°. The departures from 90° profiles are still quite small. In this case, the 90° profiles could be used for angles from 75° to 105°. If the solutions had been obtained for surface inclinations greater than 120°, the effect of the lateral buoyancy force should become important again. Then a comparison of the profiles for inclinations on opposite sides of the vertical should have revealed a thinner boundary layer on the downward-facing side, that is for $\alpha > 90°$. However, difficulties were encountered in starting the numerical procedure and the solution attempts had to be abandoned. The difficulties were attributed to the increasing importance of the lateral buoyancy force, which indirectly causes a positive longitudinal pressure gradient which opposes a flow in the positive ξ-direction. If the longitudinal buoyancy force is not of sufficient strength to overcome this positive pressure gradient, then it is impossible for a boundary layer to develop. It has been known for some time that a boundary-layer flow on a downward-facing, horizontal, isothermally-heated surface ($\alpha = 180°$) is not possible [18, 19]. However, it is not known if the theory will predict the break-down of the boundary-layer formulation at some inclination less than 180°. The difficulties encountered in attempting to generate solutions for inclinations greater than 120° would seem to indicate that such a limiting

inclination may exist, but unfortunately the numerical procedure appears to be incapable of finding this limit.

Figures 4 and 5 indicate the development of the temperature, pressure and velocity profiles at various positions along the surface for surface inclinations of 45° and 75°, respectively. It is evident from these profiles that the thermal and momentum boundary layers continue to grow with distance from the leading edge. This is in contradiction to an assumption used in obtaining the disturbance equations 2.3-6 and 2.3-13. However, since the neglected terms are all of the order of $Ra_X^{-\frac{1}{5}(1+\omega^2)}$ or less relative to the largest terms retained, this contradiction may have little bearing on the stability results.

It has been stated previously that the boundary-layer problem posed for the present analysis is non-similar except for the surface inclinations of 0° and 90°. The departure from similarity for 45° and 75° are indicated in figures 6 and 7, respectively. Although the magnitudes of the departures indicated are a reflection of the particular function $H(\omega)$ used in the order-of-magnitude analysis and the particular transformation applied to the equations, these figures do indicate qualitatively the departures that would result for any other choice of $H(\omega)$, $d(\omega)$ and $h(\omega)$ indicated in appendices A, B and C, respectively. The profiles for $\xi = 0.00625$ do not appear to fit the family of curves indicated by the profiles for the other positions. This may be attributed, at least in part, to the non-zero value of ξ used in the lateral pressure gradient term to start the integration procedure as mentioned in section 3.1. However, as mentioned in section 4.1-1, the numerical experiments with $\xi°$ indicated that this effect was not significant for most of the other values of ξ.

4.1-4 Comparison with Previous Theoretical and Experimental Work

The available information for free-convection flows over inclined-surfaces is still somewhat sparse. The earliest work on the inclined-surface problem appears to be that of Rich [22]. Rich's experimental work covered a range of angles from the vertical to 40° from the vertical with the heated surface facing up. For inclinations further from the vertical, the flow became three-dimensional. Hassan and Mohammed [26] attempted to explain the three-dimensional effect observed by Rich as a separation of the flow. However, since Rich's experiments were for large Grashof numbers and since he observed fluctuations in his results, the phenomenon might also be explained as the formation of a set of longitudinal roll vortices of the form observed by Sparrow and Husar [44]. The experimental results obtained by Rich were later found to be in good agreement with the theoretical and experimental results of Kierkus [25] and with the experimental results of Hassan and Mohammed [26].

Levy [23] used integral methods to study various examples of free-convection flow. His predictions for the heat transfer were found to be in good agreement with the experimental results of Rich. Levy's heat transfer results for a vertical and a horizontal isothermal plate are indicated in figures 8 and 9. These figures compare the effects of inclination on the heat transfer results as predicted by the present analysis with the results of previous theoretical and experimental investigations for $\xi = 0.2$ and for $\xi = 1.0$. The Nusselt numbers predicted by the present analysis are slightly higher than Levy's second approximations for the horizontal. For the vertical plate, Levy's second approximations are

much higher than the present results. However, using a quartic, Levy's results for the vertical plate are very close to the present results.

The theoetical heat transfer results of Stewartson [18] and Ostrach [3] are also indicated in figures 8 and 9. The present results are in good agreement with these results. However, since the transformation given in appendix B is based on and can be reduced to the similarity transformations for 0° and 90°, the results of the present analysis for 0° and 90° would be expected to agree with the similarity solutions obtained by Stewartson and Ostrach, respectively.

Michiyoshi's heat transfer results [24] are also shown in figures 8 and 9. The agreement between the present results and Michiyoshi's results is quite good over the range of inclinations from 30° to 120° except near 90° for ξ=0.2. This effect may be due to the curvature of Michiyoshi's plate near the leading edge. The poor agreement between 0° and 30° is probably due to the neglect of the longitudinal pressure gradient in Michiyoshi's work. Through its coupling with the lateral pressure gradient and the lateral buoyancy force, this term is very important for 0° since it provides the only driving force for a flow. For ξ=1.0, the zero heat transfer for 0° in Michiyoshi's work can be explained in the following manner. This position is the mid-point along the upper surface of the plate. Since the plate is finite, the flow starts at both ends of the plate and proceeds towards the mid-point of the plate, where the flow rises in a plume. Because of the formation of a plume, there is no flow parallel to the surface at ξ=1.0. Therefore, from Michiyoshi's equation (6), it can be seen that the local heat transfer is zero. However, if the surface is semi-infinite as in the present analysis, then the flow starts at the leading edge and proceeds

along the surface without the formation of a plume.

The heat transfer results obtained by Kierkus [25] using a perturbation analysis are also given in figures 8 and 9. Over the range of inclinations for which a comparison is possible, the results of the present analysis are higher than Kierkus' results, but the results are still in good agreement over a range of inclinations within about 15° of the vertical. However, the difference between the results increases markedly as the inclination goes beyond 15° to 20° from the vertical, especially for $\xi=0.2$. Since a perturbation analysis is limited by the size of the perturbation parameter, the increasing difference between the results can be explained at least partially by noting that Kierkus' analysis is approaching its limit of validity. A further comparison between the present results and Kierkus' theoretical results is given in table 3, in which the surface pressures are compared and found to be in good agreement.

An interesting point about Kierkus' results is that for the vertical they do not agree with the vertical solutions used as his zeroth-order approximation (although Kierkus apparently intended this to be the case). Although an examination of Kierkus' equation (20) reveals that it is clearly non-homogeneous except for a vertical plate, Kierkus stated that the equation is homogeneous; on this basis, he concluded that homogeneous boundary conditions were unacceptable since they would yield only trivial solutions for the first-order approximations. Kierkus thus established a set of non-homogeneous boundary conditions following a procedure used by Yang and Jerger [52], who analysed a free-convection flow over a vertical plate using a perturbation analysis with $Gr^{-\frac{1}{4}}$ as their perturbation parameter. In their

analysis, Yang and Jerger neglected terms of the order of $Gr^{-1/2}$ or less (that is second- and higher-order perturbations). Since there were no terms in their equations having a coefficient of $Gr^{-1/4}$, all of the terms remaining in the equations were included in the zeroth-order approximations. Since the zeroth-order approximations satisfied all of the imposed boundary conditions of the problem, the zeroth-order solutions should have been the solutions to the problem at least to the order of the terms neglected. To improve the solutions, some of the neglected terms should be added; but the first-order approximations do not include any of these terms and therefore trivial solutions should be expected. Since Yang and Jerger imposed non-homogeneous boundary conditions for their first-order approximation, one is lead to believe that the boundary conditions of the original problem were somehow incorrect. If this were true, the zeroth-order approximation could have accounted for a modified boundary condition. It appears that the solutions presented by Yang and Jerger and by Kierkus are incorrect.

Yang and Jerger stated that the boundary-layer solutions must match the potential-flow solutions and that one of the boundary conditions for their potential flow is given by their equation (21). In addition, they stated* that $U^{(1)}$ and $V^{(1)}$ must tend to zero as y tends to infinity. Since the potential-flow equations are first order in y for both $U^{(1)}$ and $V^{(1)}$, then $U^{(1)}$ and $V^{(1)}$ can each satisfy either a condition at $y = 0$ or a condition for $y \to \infty$ but not both. If $U^{(1)} = 0$ as $y \to \infty$ is imposed, then it would be fortuitous if $U^{(1)}$ at $y = 0$ matched the boundary-layer solution. Yang and Jerger, and later Kierkus, followed this approach and found that

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* Their notation is used here.

$U^{(1)}$ did not match the boundary-layer solution. They modified the first-order boundary-layer solution accordingly, but they also imply that $V^{(1)}$ matches the zeroth-order boundary-layer solution before modification and that $V^{(1)} = 0$ as $y \to \infty$. This amounts to imposing more conditions on $V^{(1)}$ than the equations can satisfy. Therefore, it is felt that Yang and Jerger and Kierkus were in error in assuming that the boundary-layer solution follows from the potential-flow solution rather than vice versa, and consequently their results must be in error.

In addition to his theoretical work, Kierkus performed some experiments on the free-convection flow about an inclined isothermal plate. He used a Mach-Zehnder interferometer to measure the velocity and temperature profiles. Hassan and Mohamed [26] also conducted an extensive experimental investigation of the heat transfer from an inclined, isothermal flat surface for a wide range of inclinations which corresponds to 0° to 180°. They used Boelter-Schmidt heat flux meters to determine the local heat-transfer coefficient along the surface, and they found that most of their heat transfer data could be correlated within ±10% by the relation

$$Nu_x = 0.348\ (Gr_x \sin\alpha)^{1/4}.$$

Using this relation, some experimental points are indicated in figures 8, 9, 10 and 11. These figures reveal that the present theoretical results are well within ten per cent of the above relation, thus indicating good agreement between theory and experiment. The theoretical results of Kierkus appear to be in better agreement with the experimental results, but in view of the above discussion of Kierkus' work, this conclusion may be unjustified. However, figures 8 and 9 do indicate that the present results are in better agreement with Hassan and Mohamed's experimental

results than are Michiyoshi's results.

Figures 10 and 11 indicate the variation with position along the surface of the heat transfer for 45° and 75°, respectively. Some experimental results of Hassan and Mohamed are also shown. The agreement between theory and experiment is again very good. Kierkus' theoretical results are also indicated in figures 10 and 11, but again the apparent good agreement of his results with the present results and with those of Hassan and Mohamed may be fortuitous.

4.2 Disturbance Flow

4.2-1 Errors and Results

Solutions of the disturbance equations have also been obtained for air (Prandtl number of 0.72) adjacent to a flat, two-dimensional, isothermal surface. For the Tollmien-Schlichting wave disturbances, equations 2.3-6 subject to the boundary conditions 2.3-7 and 2.3-8 have been solved under the conditions specified above. For the Taylor-Goertler roll-vortex disturbances, solutions have been obtained for equations 2.3-13 with the boundary conditions 2.3-14. All of the solutions were generated on an IBM 360/67 computer at the University of Alberta. The computations were done in single precision because it was found that a change to double precision resulted in a change in the eigenvalues in or beyond the sixth significant figure. The values for Δy and y_e were determined by the values used in generating the mean-flow solutions. To obtain solutions of the disturbance equations at Rayleigh numbers for which there were no mean-flow results, the mean-flow profiles were interpolated using a computer-supplied routine which uses Chebyshev polynomials. For input parameters, this routine requires bounds on the relative loss of significance

and the relative error in the interpolation. These bounds were specified as 10^{-10} and 10^{-8}, respectively.

Figure 12 presents the neutral stability curves for the Tollmien-Schlichting wave disturbances for surface inclinations of 30°, 37.5°, 45°, 60°, 75°, 90°, 105° and 120°. Most of the curves have been generated for Rayleigh numbers up to 10^6, but the 90° curves have been extended to Rayleigh numbers of 2.8×10^6 on the upper branch and 4.0×10^6 on the lower branch. Figure 12a presents the neutral stability curves in terms of the wavenumber versus the Rayleigh number, and figure 12b presents the neutral stability curves in terms of the frequency versus the Rayleigh number. The curves are not extended to inclinations less than 30° because the boundary-layer assumptions are no longer valid in the neighborhood of the critical Rayleigh number. The curves do not extend beyond 120° for reasons noted in section 4.1.

Figure 13 presents the neutral stability curves for the Taylor-Goertler roll-vortex disturbances for surface inclinations of 30°, 37.5°, 45°, 60°, 75° and 85°. These curves indicate the variation of the wavenumber with Rayleigh number for each surface inclination. The results are presented for Rayleigh numbers up to 10^6 or as far as convergence could be obtained. Inclinations at or beyond 90° are not considered because this mode of instability is not possible. Some of the eigenvalues obtained for each form of instability are tabulated in tables 4 and 5.

4.2-2 Discussion of the Results

The neutral stability curves for the Tollmien-Schlichting wave disturbances are indicated in figures 12a and 12b, and these figures are interpreted in the following manner. For small Rayleigh-number flows, there are no wavenumber-frequency combinations corresponding to a boundary-

layer mean flow for which the wave disturbances will grow with time or position. As X increases, a position is reached for which there is a single wavenumber-frequency pair that produces a neutrally stable disturbance. For Rayleigh numbers above this "critical" Rayleigh number, there are ranges of wavenumbers and frequencies for which the resulting disturbances grow with position or time or both. These ranges are bounded by the upper and lower branches of the curves shown in figure 12. The upper branches of the curves in figure 12a correspond to the upper branches of the curves in figure 12b. For any inclination, any disturbances having wavenumber-frequency pairs which lie above the upper branches or below the lower branches of the curves in figures 12a and 12b are stable since they will attenuate with position or time or both.

Figures 12a and 12b reveal that the surface inclination has a very significant effect on the neutral stability curves for the Tollmien-Schlichting wave disturbances. Since the solution for 90° was previously determined by Nachtsheim [33], the 90°-curves will be used as a base from which any changes are observed. The 90°-curves are characterized by the existence of two "noses" and these will be referred to as the upper and lower noses. Nachtsheim's work revealed that in the region of the upper nose instability in the flow is associated with hydrodynamic effects in which the Reynolds stresses play a dominant role in transferring energy from the mean flow to the disturbances. In the region of the lower nose, the Reynolds stresses were found to be subtracting energy from the disturbances, and the instability resulted from an energy transfer to the disturbances by the buoyancy effects which give rise to the temperature disturbances. Hence, this latter instability is characteristic of a thermal instability. From 90° to 30°, the lower nose becomes more

prominent and the region associated with the upper nose is almost completely absorbed. This implies that the thermal instability definitely assumes the dominant role as the surface is inclined further from the vertical. However, from 90° to 120°, the buoyancy effect is in the opposite direction to that for angles between 0° and 90°, and therefore it provides a stabilizing effect on the flow.

The initial decrease in the maximum wavenumber and the maximum frequency as the inclination varies from 90° to 75° may be attributed to the decreasing role of the hydrodynamic effects noted above. Since the energy transfer associated with the Reynolds stresses is expected to continue to decrease with increasing inclination from the vertical, the subsequent increases in these maximum values between 75° and 30° are presumably due to the increasing importance of the buoyancy effects. In figure 12a, there is no obvious explanation for the peaks in the upper portion of the curves for 37.5° and 30°. Since this peak is slightly more prominant at 37.5°, it might be attributed to a small energy transfer associated with the Reynolds stresses, but the position of the peak does not appear to fit the pattern established for the other inclinations.

In conjunction with the increasing effect of thermal instability as the surface inclination varies from 90° to 30°, there is a very significant increase in the range of wavenumbers associated with unstable flows. This wide range of wavenumbers exists even in the region close to the critical Rayleigh number. However, for inclinations of 45° or less, the ranges of frequencies associated with unstable flows decrease in the region close to the critical Rayleigh number.

Figures 12a and 12b also reveal that the critical wavenumber increases as the inclination varies from 90° to 37.5°, but it decreases

slightly between 37.5° and 30° as well as from 90° to 120°. It is not apparent from the present analysis why, or if, the critical wavenumber should reach a maximum in the vicinity of 37.5°. In contrast to the critical wavenumber, the critical frequency is very close to a maximum at 90°. It decreases at 105° and 120°, but it is nearly constant from 90° to 60° with it decreasing markedly from 60° to 30°. The critical frequency at 75° appears to be slightly higher than the value at 90° but it is difficult to determine whether or not this is due to small numerical errors.

The neutral stability curves for 30°, 37.5°, 45° and 60° indicate possible closure of the curves as the Rayleigh number tends to infinity. However, the profiles have not been extended far enough to draw any definite conclusion on this point. There is a possibility that the wavenumbers associated with the upper branch may tend to oscillate as the Rayleigh number becomes very large. This oscillatory behaviour of the upper branch was observed by Hieber and Gebhart [40] in their stability analysis of a free-convection boundary-layer flow over a vertical uniform-heat-flux plate. Hieber and Gebhart were unable to explain this phenomenon and there is nothing in the present analysis to indicate why such a behaviour might be observed.

Figure 13 presents the neutral stability curves for the Taylor-Goertler roll-vortex disturbances in terms of the wavenumber versus the Rayleigh number based on the position along the surface. Unlike the neutral stability curves for the wave disturbances, the curves in figure 13 do not exhibit any significant change in shape with a change in the surface inclination. These curves again separate the stable and unstable regions, but these regions are much more simply defined than for the wave

disturbances. For a given Rayleigh number and a particular inclination, any set of vortices having a wavenumber above the neutral curve will be damped out with time; but this set of vortices would be amplified if its wavenumber was below the curves in figure 13 or if the Rayleigh number is increased. The lower portion of each of the curves has been obtained by extrapolation using a five-point Adam's scheme because problems were encountered with the convergences of the numerical procedure in this region.

The curves in figure 13 clearly show that the effect of the surface inclination is much greater as the inclination approaches the vertical. This can be seen by noting the variation in the wavenumber as the inclination changes for a fixed Rayleigh number. Another interesting feature of the curves in figure 13 is that each curve appears to be tending toward its own upper bound on the wavenumber. This reveals that for each surface inclination there is some size of the roll vortices below which the disturbances will not grow. This size increases as the surface inclination tends toward 90°.

Another point which requires some consideration is the meaning of the critical Rayleigh number associated with the roll-vortex disturbances. Following the definition for the wave disturbances, the critical Rayleigh number is defined as the lowest Rayleigh number at which amplification for at least one wavenumber just begins. For each of the curves shown in figure 13, the critical Rayleigh number is found by the intersection of the extrapolated neutral stability curve with the zero wavenumber line. This implies that the roll vortices associated with the critical Rayleigh number are infinitely large. This is physically impossible, and it implies that the critical Rayleigh numbers obtained

in figure 13 are actually lower bounds.

There is one other point regarding the solution of the disturbance equations which requires some consideration. In the solutions obtained for both disturbance forms, it was assumed that the mean flow could be approximated as a parallel flow. However, in section 4.1-3 it was noted that the flow is in fact non-parallel. It should be noted that some of the terms neglected by the parallel-flow assumption are of the same order as some of the terms retained in the equations. However, those terms retained involve the highest-order derivatives and the neglect of these terms would completely change the nature of the disturbance equations. The non-parallel effects are related to the inertia effects in the disturbance equations and if the inertia effects are considered by themselves then the terms neglected are of the order of $Ra_X^{-\frac{1}{5}(1+\omega^2)}$, or less, relative to the terms retained. Furthermore, since the non-parallel effects are not associated with the highest-order derivatives, it is reasonable to assume that these terms can be neglected without a significant change in any important charateristics of the disturbance equations.

4.2-3 Comparison with Previous Theoretical and Experimental Work

The critical Rayleigh numbers predicted by each of the disturbance theories used in the present analysis are plotted versus the surface inclination in figure 14. Also shown is the theoretical critical Rayleigh number for a vertical, isothermal plate obtained by Nachtsheim. The experimental results of Lloyd and Sparrow [43] are also indicated. Nachtsheim's result [33] provides the only theoretical check on the present results for the wave disturbances. Nachtsheim used a different ordering analysis to obtain the set of disturbance equations for his

stability analysis, but the equations used in the present analysis have all the terms involved in Nachtsheim's equations, only the coefficients have changed. Since the sets of equations would be identical if the variables had been normalized by the same procedure, then the results obtained by both sets must agree within the numerical accuracy when presented in the same coordinate system. Figure 14 reveals that this agreement has been achieved. The neutral stability curve for 90° in figure 12a also agrees with the neutral stability curve given by Nachtsheim. This agreement indicates that the numerical procedure for the wave disturbances is capable of giving sufficiently accurate results at least for 90°. Since this procedure is unaltered for other surface inclinations, there is no reason to doubt the accuracy of the numerical results for the other surface inclinations. In fact, since the numerical procedure used for the roll-vortex disturbances is essentially the same as that for the wave disturbances, the above agreement is considered to be an indication of the accuracy of the results obtained for the roll-vortex disturbances as well.

A comparison of the two disturbance theories reveals a very abrupt change in the form of the instability between 85° and 90° with another possible change between 15° and 20°. Since the latter change would occur at a Rayleigh number well below the limit of the boundary-layer approximations and since it has been obtained by extrapolation, this result has very little significance. In contrast, the former change is very significant since it implies a fundamental change in the mechanism of the instability. For surface inclinations of 90° or above, the present theory predicts that any instability in the flow initially

results from the formation of Tollmien-Schlichting wave disturbances which are two-dimensional. For surface inclinations of 85° or below, any instability is due initially to the formation of Taylor-Goertler roll vortices which make the flow three-dimensional.

The change from one form of instability to the other at a surface inclination between 85° and 90° does not agree with the experimental observations of Lloyd and Sparrow [43] which revealed a change between 73°and 76°. This difference is not surprising in view of the difference in the critical Rayleigh numbers predicted by theory and observed by experiment. However, in view of the work of Dring and Gebhart [36, 37] and Gebhart [53], the results observed by experiments are the final stages of the transition whereas the theoretical results are predicting the initial step towards the transition to turbulence. For a free-convection boundary-layer flow over a uniform-heat-flux vertical surface, Dring and Gebhart [37] showed that the disturbances which begin to amplify first do not have the highest amplification rate. Gebhart [53] found that the frequencies associated with the disturbances having the highest amplification rate were in good agreement with the experimental values observed by Eckert and Soehnghen [28], Szewczyk [32] and Lock, Gort and Pond [42]. This agreement suggests that the difference between the theoretical and experimental critical Rayleigh numbers can be attributed to an amplification process.

All of the above experiments were performed without introducing controlled disturbances. However, using controlled disturbances introduced by a vibrating ribbon and using a Mach Zehnder interferometer to observe these disturbances, Polymeropoulos and Gebhart [35] performed

experiments on the stability of a free-convection boundary-layer flow over a uniform-heat-flux vertical plate to determine the neutral stability curve. The results of their investigation give strong support to the linear stability theory in predicting the initial instability. Hieber and Gebhart [39,40] found empirical correlations between the linear stability theory and the experimental observations of : a)the first noticeable oscillation in the boundary-layer flow, b)the first significant departure from a laminar flow.

Although the above results seem to adequately account for the difference between the theoretical and experimental results for the wave disturbance, the difference for the roll-vortex disturbance remains to be explained. There does not appear to be any method of intoducing sets of roll vortices of a controlled size into the flow, and therefore, only "natural" roll-vortex disturbances can be observed. The theoretical predictions shown in figure 13 indicate that the flow initially becomes unstable due to the formation of very large roll vortices. In actual flows, it might be expected that the first roll vortices to form would be relatively small. Assuming that this is what actually happens, then the Rayleigh numbers indicated by the intersections of the experimental wavenumbers with the neutral stability curves in figure 13 could be several orders of magnitude higher than the values plotted in figure 14. However, since Lloyd and Sparrow did not determine the wavenumbers of the roll vortices first observed, the above explanation remains as a hypothesis.

CHAPTER V

CONCLUSIONS

The agreement between the present theoretical and the previous theoretical and experimental mean-flow results appears to justify the boundary-layer formulation of the problem as well as the transformations used in the analysis. Although the solutions presented are restricted to constant-property free-convection boundary-layer flows of air over a flat, two-dimensional, isothermal surface, it can be seen from the transformed boundary-layer equations 2.2-9 that the extension to other fluids, to more general two-dimensional surfaces and to more general surface-temperature distributions is straightforward. As pointed out in chapter 2, it is also quite simple to extend the transformed boundary-layer equations to variable-property flows if the property variations are specified. The extension to axisymmetic flows is only slightly more difficult since it requires a reformulation of the Falkner-Skan and Lefevre transformations in the manner indicated in chapter 2. Therefore, using a formulation of the problem similar to that presented in chapter 2 and using a numerical procedure such as that in chapter 3, it should be possible to obtain the solutions to a large number of steady, laminar free-convection flows of the boundary-layer type or of the buoyant-jet type.

The present solutions reveal that the surface inclination has a significant effect on the mean flow. However, the greatest effect of the inclination can be seen over the range from 0° to 60°. Within a range of about 30° on either side of the vertical, there is only a very small change in the velocity and temperature profiles. The profiles at

angles on opposite sides of the vertical are almost identical over this range. Over the same range of inclinations, the pressure profiles show a considerable change, including a change of sign as the inclination passes through 90°; but it is apparent from the velocity and temperature profiles that the pressure variation over the boundary-layer has a negligible influence on the flow at least over this range of surface inclinations within 30° of the vertical.

In contrast to the slight change in the mean profiles over the range of surface inclinations from 60° to 120°, the stability of the flow is very strongly influenced by the surface inclination over this range as reflected by the change of five to six orders of magnitude in the critical Rayleigh number. Furthermore, the mechanism of the initial instability undergoes a change from the formation of a set of Taylor-Goertler roll-vortex disturbances to the formation of Tollmien-Schlichting wave disturbances as the surface inclination varies from below to above about 85°.

For surface inclinations below approximately 80°, the present stability analysis reveals that any free-convection boundary-layer flow, which satisfies the conditions assumed in the present analysis, is potentially unstable if a set of sufficiently large roll vortices is superimposed on it. The critical size of these roll vortices decreases as the angle decreases. For surface inclinations below about 35°, the same flow is also potentially unstable if certain wave disturbances are introduced into the flow. The range of wavenumbers for unstable disturbances increases as the angle decreases. Therefore, assuming that the flow is in a region in which the boundary-layer approximations are valid, it may be concluded that for inclinations below about 80° the

flow is not completely stable to all Tollmien-Schlichting wave disturbances and all Taylor-Goertler roll-vortex disturbances. However, for inclinations greater than 80°, there is at least some portion of the boundary-layer regime over which the flow is completely stable to all sizes of both forms of the assumed disturbances.

The lack of theoretical and experimental stability analyses with which a comparison can be made makes it difficult to draw any definite conclusions regarding the validity of the present results. However, the results do appear to be reasonable in terms of the previously available results for 90°. In view of this apparent success in the present stability analysis in addition to the capability of generating the mean-flow solutions, it is now possible to extend the stability analysis to a much wider class of problems. Perhaps the first extension to the present work would be to determine the amplification rates and attempt to correlate the results to the experimental results for both disturbance forms following procedures such as those used by Dring and Gebhart [37] and by Hieber and Gebhart [39,40]. Before proceeding with other classes of problems, the influence on the stability of the non-parallel effects of the mean flow should be determined even though the effect is expected to be small. To do this, the disturbance equations 2.3-6 and 2.3-13 would have to be replaced by equations 2.3-5 and 2.3-12, respectively, but the numerical procedure should not require any modification for the wave disturbances. However, the procedure for the roll-vortex disturbances has to be modified to solve for four linearly independent solutions because of the additional coupling related to δ_1. Depending on the outcome of this investigation, either equations 2.3-5 and 2.3-12 or equations 2.3-6

and 2.3-13 could then be used to analyse the stability of constant-property free-convection boundary-layer flows of different fluids over more general two-dimensional surfaces having more general surface-temperature distributions.

In order to consider the stability of variable-property or axisymmetric flows, the more general disturbance equations 2.3-3 and 2.3-10 can be considered for the wave disturbances and the roll-vortex disturbances, respectively. Although these equations are much more complicated, the procedure for solving these equations should be basically the same as the procedure used in the present work. Obtaining the asymptotic solutions may be slightly more complicated, but once these are obtained, the procedure for obtaining the linearly independent solutions should be straightforward. For the wave disturbances, there will be three equations to solve instead of two because the disturbance stream function cannot be introduced, but again this should not present any serious difficulties. Therefore, within the limitations of the linear stability theory, the formulation of the equations and the procedure for solving the equations appears to be capable of analysing the stability of a large class of free-convection boundary-layer of buoyant jet flows.

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Appendix A - A Normalization and an Order-of-Magnitude Analysis of the Mean-Flow Equations

As a first step in seeking any simplification to the set of steady, two-dimensional equations of motion 2.1-2, an attempt is made to normalize the dependent and independent variables by introducing a set of characteristic quantities. The definition of each of these characteristic quantities must come from either the boundary conditions, the equations or the physical description of the problem. Introducing the normalized variables and the characteristic quantities into equations 2.1-2 yields the following set of equations:

$$[\frac{\rho_c U_c}{X_c}][\frac{\gamma\partial u}{\partial x} + \frac{u\partial\gamma}{\partial x} + \frac{\gamma u}{(j\pm Jysin\alpha)}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx})] + [\frac{\rho_c V_c}{Y_c}][(1+Aqy)(\frac{\gamma\partial v}{\partial y} + \frac{v\partial\gamma}{\partial y})$$

$$+ Aqv \pm \frac{vJsin\alpha}{(j\pm Jysin\alpha)}] = 0$$

$$[\frac{\rho_c U_c^2}{X_c}]\frac{\gamma u}{(1+Aqy)}\frac{\partial u}{\partial x} + [\frac{\rho_c V_c U_c}{Y_c}][\frac{\gamma v\partial u}{\partial y} + \frac{Aq\gamma uv}{(1+Aqy)}] = -[\frac{P_c}{X_c}]\frac{1}{(1+Aqy)}\frac{\partial P_d}{\partial x}$$

$$+ [\frac{\mu_c U_c}{X_c^2}]\frac{1}{(1+Aqy)^2}\{(\frac{\acute{M}+4M}{3})[\frac{\partial^2 u}{\partial x^2} + \frac{1}{(j\pm Jysin\alpha)}(\frac{ud^2j}{dx^2} \mp uJysin\alpha(\frac{d\alpha}{dx})^2$$

$$\pm uJycos\alpha\frac{d^2\alpha}{dx^2} + \frac{\partial u}{\partial x}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx})) - \frac{u}{(j\pm Jysin\alpha)^2}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx})^2$$

$$- \frac{Ay}{(1+Aqy)}\frac{dq}{dx}(\frac{\partial u}{\partial x} + \frac{u}{(j\pm Jysin\alpha)}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}))] + (\frac{\partial\acute{M}}{\partial x} + \frac{4}{3}\frac{\partial M}{\partial x})\frac{\partial u}{\partial x}$$

$$+ \frac{u}{(j \pm Jysin\alpha)}(\frac{\partial \acute{M}}{\partial x} - \frac{2}{3}\frac{\partial M}{\partial x})(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx})\} + [\frac{\mu_c V_c}{X_c Y_c}]\frac{1}{(1+Aqy)}\{(\frac{\acute{M}+M}{3})[\frac{\partial^2 v}{\partial x \partial y}$$

$$\pm \frac{Jsin\alpha}{(j \pm Jysin\alpha)}\frac{\partial v}{\partial x}] + (\frac{\acute{M}+4M}{3})\frac{Jvcos\alpha}{(j \pm Jysin\alpha)}\frac{d\alpha}{dx} + \frac{(\acute{M}+7/3M)Aq}{(1+Aqy)}\frac{\partial v}{\partial x}$$

$$+ \frac{(\acute{M}+4/3M)}{(1+Aqy)}[Av\frac{dq}{dx} \mp \frac{Jvsin\alpha}{(j \pm Jysin\alpha)}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}) - \frac{A^2yqv}{(1+Aqy)}\frac{dq}{dx}]$$

$$+ (\frac{\partial \acute{M}}{\partial x} + \frac{4}{3}\frac{\partial M}{\partial x})\frac{Aqv}{(1+Aqy)} + (\frac{\partial \acute{M}}{\partial x} - \frac{2}{3}\frac{\partial M}{\partial x})(\frac{\partial v}{\partial y} \pm \frac{Jvsin\alpha}{(j \pm Jysin\alpha)}) + \frac{\partial M}{\partial y}\frac{\partial v}{\partial x}\}$$

$$+ [\frac{\mu_c U_c}{Y_c^2}]\{\frac{\partial M}{\partial y}(\frac{\partial u}{\partial y} - Aqu) + M[\frac{\partial^2 u}{\partial y^2} \pm \frac{Jsin\alpha}{(j \pm Jysin\alpha)}\frac{\partial u}{\partial y}] + \frac{MAq}{(1+Aqy)}[\frac{\partial u}{\partial y}$$

$$\pm \frac{Jusin\alpha}{(j \pm Jysin\alpha)}] - \frac{MA^2q^2u}{(1+Aqy)^2}\} - [\rho_c\beta\Theta_c g]\frac{\gamma_d}{\beta\Theta_c}\sin\alpha \tag{A-1}$$

$$[\frac{\rho_c U_c V_c}{X_c}]\frac{\gamma u}{(1+Aqy)}\frac{\partial v}{\partial x} + [\frac{\rho_c V_c^2}{Y_c}]\gamma v\frac{\partial v}{\partial y} - [\frac{\rho_c U_c^2}{Y_c}]\frac{\gamma Aqu^2}{(1+Aqy)} = -[\frac{P_c}{Y_c}]\frac{\partial p_d}{\partial y}$$

$$+ [\frac{\mu_c V_c}{Y_c^2}]\{(\frac{\acute{M}+4M}{3})[\frac{\partial^2 v}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial v}{\partial y} \pm \frac{Jsin\alpha}{(j \pm Jysin\alpha)}\frac{\partial v}{\partial y} \mp \frac{J^2 vsin^2\alpha}{(j \pm Jysin\alpha)^2}$$

$$- \frac{A^2q^2v}{(1+Aqy)^2}] + (\frac{\partial \acute{M}}{\partial y} + \frac{4}{3}\frac{\partial M}{\partial y})\frac{\partial v}{\partial y} + (\frac{\partial \acute{M}}{\partial y} - \frac{2}{3}\frac{\partial M}{\partial y})[Aqv \pm \frac{Jvsin\alpha}{(j \pm Jysin\alpha)}]\}$$

$$+ [\frac{\mu_c U_c}{X_c Y_c}]\frac{1}{(1+Aqy)}\{(\frac{\acute{M}+M}{3})\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial M}{\partial x}\frac{\partial u}{\partial y} + (\frac{\partial \acute{M}}{\partial y} - \frac{2}{3}\frac{\partial M}{\partial y})[\frac{\partial u}{\partial x} + \frac{(1+Aqy)u}{(j \pm Jysin\alpha)}(\frac{dj}{dx}$$

$$\pm \, Jy\cos\alpha\frac{d\alpha}{dx})] + \frac{1}{(j\pm Jy\sin\alpha)}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})[(\frac{\acute{M}+M}{3})\frac{\partial u}{\partial y} - \frac{AquM}{(1+Aqy)}$$

$$\mp \frac{Ju\sin\alpha}{(j\pm Jy\sin\alpha)}(\acute{M} + \frac{4M}{3}) - \frac{(\acute{M}+4/3M)Aqu}{(1+Aqy)}] + (\acute{M} + \frac{4M}{3})[\frac{\pm Ju\cos\alpha}{(j\pm Jy\sin\alpha)}\frac{d\alpha}{dx}$$

$$- \frac{Aq}{(1+Aqy)}\frac{\partial u}{\partial x}] - \frac{Aqu}{(1+Aqy)}\frac{\partial M}{\partial x} - \frac{MA}{(1+Aqy)}[\frac{udq}{dx} - \frac{q\partial u}{\partial x} - \frac{Aqyu}{(1+Aqy)}\frac{dq}{dx}]\}$$

$$+ [\frac{\mu_c V_c}{X_c^2}]\frac{1}{(1+Aqy)^2}[\frac{\partial M}{\partial x}\frac{\partial v}{\partial x} + \frac{M\partial^2 v}{\partial x^2} + \frac{M}{(j\pm Jy\sin\alpha)}\frac{\partial v}{\partial x}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})$$

$$- \frac{MAy}{(1+Aqy)}\frac{dq}{dx}\frac{\partial v}{\partial x}] - [\rho_c\beta\Theta_c g]\frac{\gamma_d}{\beta\Theta_c}\cos\alpha$$

$$[\frac{\rho_c C_c U_c \Theta_c}{X_c}]\frac{\gamma cu}{(1+Aqy)}[(1+a)\frac{\partial\theta}{\partial x} + \frac{\theta da}{dx}] + [\frac{\rho_c C_c V_c \Theta_c}{Y_c}]\gamma cv(1+a)\frac{\partial\theta}{\partial y}$$

$$+ \frac{T}{\rho}(\frac{\partial\rho}{\partial T})_P\{[\frac{U_c P_c}{X_c}]\frac{u}{(1+Aqy)}\frac{\partial p}{\partial x} + [\frac{V_c P_c}{Y_c}]\frac{v\partial p}{\partial y}\}$$

$$= [\frac{k_c\Theta_c}{X_c^2}]\frac{1}{(1+Aqy)^2}\{K(1+a)\frac{\partial^2\theta}{\partial x^2} + \frac{2K\partial\theta}{\partial x}\frac{da}{dx} + \frac{K\theta d^2 a}{dx^2} + [\frac{K}{(j\pm Jy\sin\alpha)}(\frac{dj}{dx}$$

$$\pm \, Jy\cos\alpha\frac{d\alpha}{dx}) + \frac{\partial K}{\partial x} - \frac{KAy}{(1+Aqy)}\frac{dq}{dx}][(1+a)\frac{\partial\theta}{\partial x} + \frac{\theta da}{dx}]\} + [\frac{k_c\Theta_c}{Y_c^2}][K(1+a)\frac{\partial^2\theta}{\partial y^2}$$

$$+ (1+a)\frac{\partial K}{\partial y}\frac{\partial\theta}{\partial y} \pm \frac{KJ\sin\alpha(1+a)}{(j\pm Jy\sin\alpha)}\frac{\partial\theta}{\partial y} + \frac{KAq(1+a)}{(1+Aqy)}\frac{\partial\theta}{\partial y}]$$

$$+ \left[\frac{\mu_c U_c^2}{X_c^2}\right]\left\{\frac{(\acute{M}+4/3M)}{(1+Aqy)^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{u^2}{(j\pm Jysin\alpha)^2}\left(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}\right)^2\right]\right.$$

$$+ \frac{2(\acute{M}-2/3M)u}{(j\pm Jysin\alpha)(1+Aqy)^2}\frac{\partial u}{\partial x}\left(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}\right)\Big\} + \left[\frac{\mu_c V_c^2}{Y_c^2}\right]\Big\{\left(\acute{M} + \frac{4M}{3}\right)\Big[\frac{A^2q^2v^2}{(1+Aqy)^2}$$

$$+ \left(\frac{\partial v}{\partial y}\right)^2 + \frac{J^2v^2sin^2\alpha}{(j\pm Jysin\alpha)^2}\Big] + \frac{2(\acute{M}-2/3M)}{(j\pm Jysin\alpha)}\Big[\pm Jsin\alpha\frac{\partial v}{\partial y} + \frac{(j\pm Jysin\alpha)}{(1+Aqy)}Aqv\frac{\partial v}{\partial y}$$

$$\pm \frac{JAqv^2sin\alpha}{(1+Aqy)}\Big]\Big\} + \left[\frac{\mu_c U_c V_c}{X_c Y_c}\right]\frac{1}{(1+Aqy)}\Big\{\left(\acute{M} + \frac{4M}{3}\right)\Big[\frac{2Aqv}{(1+Aqy)}\frac{\partial u}{\partial x}$$

$$\pm \frac{2Juvsin\alpha}{(j\pm Jysin\alpha)^2}\left(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}\right)\Big] + \frac{2(\acute{M}-2/3M)}{(j\pm Jysin\alpha)}\Big[(j\pm Jysin\alpha)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

$$\pm Jvsin\alpha\frac{\partial u}{\partial x} + \frac{u\partial v}{\partial y}\left(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}\right) + \frac{Aquv}{(1+Aqy)}\left(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx}\right)\Big]$$

$$+ \frac{2M\partial v}{\partial x}\Big[\frac{\partial u}{\partial y} - \frac{2Aqu}{(1+Aqy)}\Big]\Big\} + \left[\frac{\mu_c U_c^2}{Y_c^2}\right]\Big[M\left(\frac{\partial u}{\partial y}\right)^2 - \frac{2MAqu}{(1+Aqy)}\frac{\partial u}{\partial y} + \frac{MA^2q^2u^2}{(1+Aqy)^2}\Big]$$

$$+ \left[\frac{\mu_c V_c^2}{X_c^2}\right]\frac{M}{(1+Aqy)^2}\left(\frac{\partial v}{\partial x}\right)^2$$

where $A = \kappa_c Y_c$

and $J = \dfrac{Y_c}{R_c}$.

With the equations in the above form, it is now possible to proceed with an order-of-magnitude analysis. In this analysis it is assumed that all nondimensional terms are normalized and that the order of each group of terms in equations A-1 is determined by the set of characteristic quantities appearing in front of each group.

Considering the conservation of mass equation, or the continuity equation, there are only two groups of terms. If the order of either group is much less than the order of the other group, it can be concluded that a special relationship exists between one of the velocities, the density and possibly the curvature. Since this relationship is not general, it follows that both groups of terms are of the same order. This implies the following:

$$\frac{U_c}{X_c} = \frac{V_c}{Y_c} \, . \tag{A-2}$$

A second relationship between the characteristic quantities can be obtained by considering the energy equation. Since the fluid flow is thermally generated, it seems reasonable to expect the order of the advection terms to be equal to the order of the larger conduction terms, namely the lateral conduction terms. Equating the corresponding sets of characteristic quantities leads to the following relationship:

$$\frac{U_c}{X_c} = \frac{\nu_c}{\sigma Y_c^{\,2}} \, . \tag{A-3}$$

The remaining relationships necessary for determining the characteristic quantities can be found only by considering equations A-1 for the special cases of $\alpha = 0$ and $\alpha = \pi/2$. On the basis of the

characteristic quantities found for the special cases, an attempt is made to generalize these quantities for any angle.

Firstly, consider equations A-1 for $\alpha = 0$. For this surface orientation, the lateral body or buoyancy force in the momentum equations provides the only driving force for any subsequent motion. This lateral buoyancy force gives rise to a lateral pressure (departure) gradient, which in turn establishes a longitudinal pressure (departure) gradient. Since it is this longitudinal pressure gradient that establishes a longitudinal driving force, it follows that the lateral pressure gradient must be of the same order of importance as the lateral buoyancy force. This is expressed in the following equation:

$$\frac{P_c}{Y_c} = \rho_c \beta \Theta_c g. \tag{A-4}$$

In addition, the largest viscous terms in the longitudinal momentum equation are expected to be of the same order as the longitudinal pressure gradient, at least for fluids having moderate or high Prandtl numbers. (For low and moderate Prandtl-number fluids, the inertia terms are expected to be of the same order as the pressure gradient.) This leads to an expression of the form:

$$\frac{P_c}{X_c} = \frac{\mu_c U_c}{Y_c^{\,2}}. \tag{A-5}$$

Now a combination of equations A-2, A-3, A-4 and A-5 yields a set of characteristic quantities given by:

$$Y_c = X_c \, Ra_{X_c}^{-\frac{1}{5}},$$

$$U_c = \frac{\nu_c}{\sigma X_c} \, Ra_{X_c}^{\frac{2}{5}},$$

$$V_c = \frac{\nu_c}{\sigma X_c} \, Ra_{X_c}^{\frac{1}{5}} \qquad \text{(A-6)}$$

and $$P_c = \frac{\rho_c \nu_c^2}{\sigma X_c^4} \, Ra_{X_c}^{\frac{4}{5}},$$

where $$Ra_{X_c} = \frac{\beta \Theta_c g X_c^3 \sigma}{\nu_c^2}$$

Secondly, consider equations A-1 for $\alpha = \pi/2$. For this surface orientation, the buoyancy force in the longitudinal momentum equation provides the driving force for any subsequent motion. For fluids having moderate or high Prandtl numbers, it is assumed that the largest viscous terms in the longitudinal momentum equation are of the same order as the buoyancy force. (For low- and moderate-Prandtl-number fluids, the inertia terms are assumed to be of the same order as the buoyancy force.) The resulting expression is:

$$\frac{\mu_c U_c}{Y_c^2} = \rho_c \beta \Theta_c g \, . \qquad \text{(A-7)}$$

If it is assumed that equation A-4 applies for $\alpha=\pi/2$ as well as for $\alpha=0$, then a combination of equations A-2, A-3, A-4 and A-7 defines a set of characteristic quantities as follows:

$$Y_c = X_c Ra_{X_c}^{-\frac{1}{4}},$$

$$U_c = \frac{\nu_c}{\sigma X_c} Ra_{X_c}^{\frac{1}{2}}, \tag{A-8}$$

$$V_c = \frac{\nu_c}{\sigma X_c} Ra_{X_c}^{\frac{1}{4}}$$

and

$$P_c = \frac{\rho_c \nu_c^2}{\sigma X_c^2} Ra_{X_c}^{\frac{3}{4}}.$$

Finally, on the basis of the expressions given by A-6 and A-8, an attempt is made to extend the order-of-magnitude analysis to any angle. To accomplish this, consider one of the characteristic quantities, whose form varies with angle, and generalize this quantity by introducing an unknown function of the angle. For example, let

$$U_c = \frac{\nu_c}{\sigma X_c} Ra_{X_c}^{H(\omega)}, \tag{A-9}$$

where $H(0) = 2/5$ and $H(1/2) = 1/2$. By inserting A-9 into equations A-2, A-3 and A-4, the following results are obtained:

$$Y_c = X_c Ra_{X_c}^{-\frac{1}{2}H(\omega)},$$

$$V_c = \frac{\nu_c}{\sigma X_c} Ra_{X_c}^{\frac{1}{2}H(\omega)} \tag{A-10}$$

and

$$P_c = \frac{\rho_c \nu_c^2}{\sigma X_c^2} Ra_{X_c}^{1-\frac{1}{2}H(\omega)}.$$

If expression A-9 and A-10 are introduced into equations A-1, the following set of equations is obtained:

$$[\frac{\rho_c \nu_c}{\sigma X_c^{\ 2}} Ra_{X_c}^{\ H}]\{\frac{\gamma \partial u}{\partial x} + \frac{u \partial \gamma}{\partial x} + \frac{\gamma}{(j \pm J y \sin\alpha)}[\pm vJ\sin\alpha + u(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})]$$

$$+ (1+Aqy)(\frac{\gamma \partial v}{\partial y} + \frac{v \partial \gamma}{\partial y}) + Aq\gamma v\} = 0$$

$$[\frac{\rho_c \nu_c^{\ 2}}{\sigma^2 X_c^{\ 3}} Ra_{X_c}^{\ 2H}]\{\frac{\gamma u}{(1+Aqy)}\frac{\partial u}{\partial x} + \frac{\gamma v \partial u}{\partial y} + \frac{\gamma Aquv}{(1+Aqy)}\}$$

$$= - [\frac{\rho_c \nu_c^{\ 2}}{\sigma X_c^{\ 3}} Ra_{X_c}^{\ 1-\frac{1}{2}H}]\frac{1}{(1+Aqy)}\frac{\partial P_d}{\partial x} + [\frac{\rho_c \nu_c^{\ 2}}{\sigma X_c^{\ 3}} Ra_{X_c}^{\ H}]\frac{1}{(1+Aqy)}\{\frac{(\acute{M}+4/3M)}{(1+Aqy)}[\frac{\partial^2 u}{\partial x^2}$$

$$+ \frac{1}{(j \pm Jy\sin\alpha)}(\frac{ud^2j}{dx^2} \mp uJy\sin\alpha(\frac{d\alpha}{dx})^2 \pm uJy\cos\alpha\frac{d^2\alpha}{dx^2} + \frac{\partial u}{\partial x}\frac{dj}{dx}$$

$$\pm Jy\cos\alpha\frac{d\alpha}{dx}\frac{\partial u}{\partial x}) - \frac{u}{(j \pm Jy\sin\alpha)^2}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})^2 - \frac{Ay}{(1+Aqy)}\frac{dq}{dx}(\frac{\partial u}{\partial x}$$

$$+ \frac{u}{(j \pm Jy\sin\alpha)}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx}))] + [(\frac{\partial \acute{M}}{\partial x} + \frac{4}{3}\frac{\partial M}{\partial x})\frac{\partial u}{\partial x} + \frac{u}{(j \pm Jy\sin\alpha)}(\frac{\partial \acute{M}}{\partial x}$$

$$- \frac{2}{3}\frac{\partial M}{\partial x})(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})]/(1+Aqy) + (\acute{M} + \frac{M}{3})[\frac{\partial^2 v}{\partial x \partial y} \pm \frac{J\sin\alpha}{(j \pm Jy\sin\alpha)}\frac{\partial v}{\partial x}]$$

$$+ (\acute{M} + \frac{4M}{3})\frac{Jv\cos\alpha}{(j \pm Jy\sin\alpha)}\frac{d\alpha}{dx} + \frac{(\acute{M}+7/3M)}{(1+Aqy)}\frac{Aq\partial v}{\partial x} + \frac{(\acute{M}+4/3M)}{(1+Aqy)}[\frac{Avdq}{dx}$$

$$\mp \frac{Jv\sin\alpha}{(j \pm Jy\sin\alpha)}\left(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx}\right) - \frac{A^2yqv}{(1+Aqy)}\frac{dq}{dx}] + \left(\frac{\partial \acute{M}}{\partial x} + \frac{4}{3}\frac{\partial M}{\partial x}\right)\frac{Aqu}{(1+Aqy)}$$

$$+ \left(\frac{\partial \acute{M}}{\partial x} - \frac{2}{3}\frac{\partial M}{\partial x}\right)\left(\frac{\partial v}{\partial y} \pm \frac{Jv\sin\alpha}{(j \pm Jy\sin\alpha)}\right) + \frac{\partial M}{\partial y}\frac{\partial v}{\partial x}\} + [\frac{\rho_c \nu_c^2}{\sigma X_c^3} Ra_{X_c}^{2H}]\{\frac{\partial M}{\partial y}\left(\frac{\partial u}{\partial y}\right.$$

$$\left.- Aqu\right) + M[\frac{\partial^2 u}{\partial y^2} \pm \frac{J\sin\alpha}{(j \pm Jy\sin\alpha)}\frac{\partial u}{\partial y}] + \frac{MAq}{(1+Aqy)}[\frac{\partial u}{\partial y} \pm \frac{Ju\sin\alpha}{(j \pm Jy\sin\alpha)}]$$

$$- \frac{MA^2q^2u}{(1+Aqy)^2}\} - [\frac{\rho_c \nu_c^2}{\sigma X_c^3} Ra_{X_c}]\frac{\gamma_d}{\beta\Theta_c}\sin\alpha \qquad \text{(A-11)}$$

$$[\frac{\rho_c \nu_c^2}{\sigma^2 X_c^3} Ra_{X_c}^{\frac{3}{2}H}]\{\frac{\gamma u}{(1+Aqy)}\frac{\partial v}{\partial x} + \gamma v\frac{\partial v}{\partial y}\} - [\frac{\rho_c \nu_c^2}{\sigma^2 X_c^3} Ra_{X_c}^{\frac{5}{2}H}]\frac{\gamma Aqu^2}{(1+Aqy)}$$

$$= - [\frac{P_c}{Y_c}]\frac{\partial P_d}{\partial y} + [\frac{\rho_c \nu_c^2}{\sigma X_c^3} Ra_{X_c}^{\frac{3}{2}H}]\{\left(\acute{M} + \frac{4M}{3}\right)[\frac{\partial^2 v}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial v}{\partial y}$$

$$\pm \frac{J\sin\alpha}{(j \pm Jy\sin\alpha)}\frac{\partial v}{\partial y} \mp \frac{J^2v\sin^2\alpha}{(j \pm Jy\sin\alpha)^2} - \frac{A^2q^2v}{(1+Aqy)^2}] + \left(\frac{\partial \acute{M}}{\partial y} + \frac{4}{3}\frac{\partial M}{\partial y}\right)\frac{\partial v}{\partial y} + \left(\frac{\partial \acute{M}}{\partial y}\right.$$

$$\left.- \frac{2}{3}\frac{\partial M}{\partial y}\right)[Aqv \pm \frac{Jv\sin\alpha}{(j \pm Jy\sin\alpha)}] + [\left(\acute{M} + \frac{M}{3}\right)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial M}{\partial x}\frac{\partial u}{\partial y}]/(1+Aqy) + \left(\frac{\partial \acute{M}}{\partial y}\right.$$

$$\left.- \frac{2}{3}\frac{\partial M}{\partial y}\right)[\frac{\partial u}{\partial x} + \frac{(1+Aqy)u}{(j \pm Jy\sin\alpha)}\left(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx}\right)]/(1+Aqy) + \frac{1}{(j \pm Jy\sin\alpha)}\left(\frac{dj}{dx}\right.$$

$$\left.\pm Jy\cos\alpha\frac{d\alpha}{dx}\right)[\left(\acute{M} + \frac{M}{3}\right)\frac{\partial u}{\partial y} - \frac{MAqu}{(1+Aqy)} - \frac{(\acute{M}+4/3M)Aqu}{(1+Aqy)}$$

$$\mp \frac{(\acute{M}+4/3M)Jusin\alpha}{(j\pm Jysin\alpha)}]/(1+Aqy) + \frac{(\acute{M}+4/3M)}{(1+Aqy)}[\frac{\pm Jucos\alpha}{(j\pm Jysin\alpha)}\frac{d\alpha}{dx} - \frac{Aq}{(1+Aqy)}\frac{\partial u}{\partial x}]$$

$$- \frac{Aqu}{(1+Aqy)^2}\frac{\partial M}{\partial x} - \frac{MA}{(1+Aqy)^2}[\frac{udq}{dx} - \frac{q\partial u}{\partial x} - \frac{Aqyu}{(1+Aqy)}\frac{dq}{dx}]\}$$

$$+ [\frac{\rho_c\nu_c^2}{\sigma X_c^3} Ra_{X_c}^{\frac{1}{2}H}]\frac{1}{(1+Aqy)^2}\{\frac{\partial M}{\partial x}\frac{\partial v}{\partial x} + \frac{M\partial^2 v}{\partial x^2} + \frac{M}{(j\pm Jysin\alpha)}\frac{\partial v}{\partial x}(\frac{dj}{dx}$$

$$\pm Jycos\alpha\frac{d\alpha}{dx}) - \frac{MAy}{(1+Aqy)}\frac{dq}{dx}\frac{\partial v}{\partial x}\} - [\frac{\rho_c\nu_c^2}{\sigma X_c^3} Ra_{X_c}]\frac{\gamma_d}{\beta\Theta_c}\cos\alpha$$

$$[\frac{k_c\Theta_c}{X_c^2} Ra_{X_c}^{H}]\{\frac{\gamma cu}{(1+Aqy)}[\frac{(1+a)\partial\theta}{\partial x} + \frac{\theta da}{dx}] + \frac{\gamma cv(1+a)\partial\theta}{\partial y}\}$$

$$+ [\frac{k_c\Theta_c}{X_c^2} OsRa_{X_c}^{\frac{1}{2}H}]\frac{T}{\rho}(\frac{\partial\rho}{\partial T})_P\{\frac{u}{(1+Aqy)}\frac{\partial p}{\partial x} + \frac{v\partial p}{\partial y}\}$$

$$= [\frac{k_c\Theta_c}{X_c^2}]\frac{1}{(1+Aqy)^2}\{\frac{K(1+a)\partial^2\theta}{\partial x^2} + \frac{2K\partial\theta}{\partial x}\frac{da}{dx} + \frac{K\theta d^2 a}{dx^2} + [\frac{K}{(j\pm Jysin\alpha)}(\frac{dj}{dx}$$

$$\pm Jycos\alpha\frac{d\alpha}{dx}) + \frac{\partial K}{\partial x} - \frac{KAy}{(1+Aqy)}\frac{dq}{dx}][\frac{(1+a)\partial\theta}{\partial x} + \frac{\theta da}{dx}]\}$$

$$+ [\frac{k_c\Theta_c}{X_c^2} Ra_{X_c}^{H}](1+a)\{\frac{K\partial^2\theta}{\partial y^2} + \frac{\partial K}{\partial y}\frac{\partial\theta}{\partial y} \pm \frac{KJsin\alpha}{(j\pm Jysin\alpha)}\frac{\partial\theta}{\partial y} + \frac{KAq}{(1+Aqy)}\frac{\partial\theta}{\partial y}\}$$

$$+ [\frac{k_c\Theta_c}{X_c^2} OsRa_{X_c}^{2H-1}]\{\frac{(\acute{M}+4/3M)}{(1+Aqy)^2}[(\frac{\partial u}{\partial x})^2 + \frac{u^2}{(j\pm Jysin\alpha)^2}(\frac{dj}{dx} \pm Jycos\alpha\frac{d\alpha}{dx})^2]$$

$$+ \frac{2(\acute{M}-2/3M)u}{(j\pm Jy\sin\alpha)}\frac{\partial u}{\partial x}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})/(1+Aqy) + (\acute{M} + \frac{4M}{3})[\frac{A^2q^2v^2}{(1+Aqy)^2} + (\frac{\partial v}{\partial y})^2$$

$$+ \frac{J^2v^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}] + \frac{2(\acute{M}-2/3M)}{(j\pm Jy\sin\alpha)}[\frac{(j\pm Jy\sin\alpha)Aqv}{(1+Aqy)}\frac{\partial v}{\partial y} \pm J\sin\alpha\frac{\partial v}{\partial y}$$

$$\pm \frac{JAqv^2\sin\alpha}{(1+Aqy)}] + \frac{(\acute{M}+4/3M)}{(1+Aqy)}[\frac{2Aqv}{(1+Aqy)}\frac{\partial u}{\partial x} \pm \frac{2Juv\sin\alpha}{(j\pm Jy\sin\alpha)^2}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})]$$

$$+ \frac{2(\acute{M}-2/3M)}{(j\pm Jy\sin\alpha)}[(j\pm Jy\sin\alpha)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} \pm Jv\sin\alpha\frac{\partial u}{\partial x} + \frac{u\partial v}{\partial y}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})$$

$$+ \frac{Aquv}{(1+Aqy)}(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx})]/(1+Aqy) + \frac{2M}{(1+Aqy)}[\frac{\partial u}{\partial y} - \frac{2Aqu}{(1+Aqy)}]\}$$

$$+ [\frac{k_c\Theta_c}{X_c^{\,2}}\, OsRa_{X_c}^{\;3H-1}]\{M(\frac{\partial u}{\partial y})^2 - \frac{2MAqu}{(1+Aqy)}\frac{\partial u}{\partial y} + \frac{MA^2q^2u^2}{(1+Aqy)^2}\}$$

$$+ [\frac{k_c\Theta_c}{X_c^{\,2}}\, OsRa_{X_c}^{\;H-1}]\frac{M}{(1+Aqy)^2}(\frac{\partial v}{\partial x})^2$$

where $Os = \frac{\beta g X_c}{C_c}$ is the Ostrach number.

The function $H(\omega)$ has no restrictions on its form as yet, though it seems reasonable to expect $H(\omega)$ to be bounded by $H(0)$ and $H(1/2)$, that is $2/5 \leq H(\omega) \leq 1/2$. Furthermore, if it is assumed that terms of the order $Ra_{X_c}^{-\frac{1}{4}}$, $\sigma Ra_{X_c}^{-\frac{1}{4}}$, $A^2\sigma Ra_{X_c}^{-\frac{1}{4}}$, Os, A^2Os, $OsRa_{X_c}^{-\frac{3}{5}}$, $A^2OsRa_{X_c}^{-\frac{3}{5}}$

or less relative to the largest terms in each equation are negligible, then equations A-11 can be simplified to the following:

$$\frac{\gamma\partial u}{\partial x} + \frac{u\partial\gamma}{\partial x} + \frac{\gamma}{(j \pm Jy\sin\alpha)}\left[\pm vJ\sin\alpha + u\left(\frac{dj}{dx} \pm Jy\cos\alpha\frac{d\alpha}{dx}\right)\right] + (1+Aqy)\left(\frac{\gamma\partial v}{\partial y} + \frac{v\partial\gamma}{\partial y}\right)$$

$$+ Aq\gamma v = 0$$

$$Ra_{X_c}^{2H-1}\gamma\left[\frac{u}{(1+Aqy)}\frac{\partial u}{\partial x} + \frac{v\partial u}{\partial y} + \frac{Aquv}{(1+Aqy)}\right] = -\sigma Ra_{X_c}^{-\frac{1}{2}H}\frac{1}{(1+Aqy)}\frac{\partial P_d}{\partial x}$$

$$+ \sigma Ra_{X_c}^{2H-1}\left\{\frac{\partial M}{\partial y}\left(\frac{\partial u}{\partial y} - Aqu\right) + M\left[\frac{\partial^2 u}{\partial y^2} \pm \frac{J\sin\alpha}{(j \pm Jy\sin\alpha)}\frac{\partial u}{\partial y}\right] + \frac{MAq}{(1+Aqy)}\left[\frac{\partial u}{\partial y}\right.\right.$$

$$\left.\left.\pm \frac{Ju\sin\alpha}{(j \pm Jy\sin\alpha)}\right] - \frac{MA^2q^2u}{(1+Aqy)^2}\right\} - \frac{\sigma\gamma_d}{\beta\Theta_c}\sin\alpha + O(\sigma Ra_{X_c}^{H-1},\ A^2\sigma Ra_{X_c}^{H-1})$$

(A-12)

$$- Ra_{X_c}^{\frac{5}{2}H-1}\frac{\gamma Aqu^2}{(1+Aqy)} = -\frac{\sigma\partial P_d}{\partial y} - \frac{\sigma\gamma_d}{\beta\Theta_c}\cos\alpha + O(Ra_{X_c}^{\frac{3}{2}H-1},\ \sigma Ra_{X_c}^{\frac{3}{2}H-1},\ A^2\sigma Ra_{X_c}^{\frac{3}{2}H-1})$$

$$\frac{\gamma cu}{(1+Aqy)}\left[(1+a)\frac{\partial\theta}{\partial x} + \frac{\theta da}{dx}\right] + \gamma cv(1+a)\frac{\partial\theta}{\partial y} = (1+a)\left[\frac{K\partial^2\theta}{\partial y^2} + \frac{\partial K}{\partial y}\frac{\partial\theta}{\partial y} \pm \frac{KJ\sin\alpha}{(j \pm Jy\sin\alpha)}\frac{\partial\theta}{\partial y}\right.$$

$$\left.+ \frac{KAq}{(1+Aqy)}\frac{\partial\theta}{\partial y}\right] + O(OsRa_{X_c}^{-\frac{1}{2}H},\ Ra_{X_c}^{-H},\ OsRa_{X_c}^{2H-1},\ A^2OsRa_{X_c}^{2H-1}, ARa_{X_c}^{-H})$$

The above analysis does not require $H(\omega)$ to take any special

form but it is expected to be a monotone function which must satisfy $2/5 \leq H(\omega) \leq 1/2$. However, in order to obtain a numerical solution, a specific form must be provided for $H(\omega)$. For this purpose, a reasonably simple, but arbitrarily chosen form is given by:

$$H(\omega) = \frac{2}{5}\,(1+\omega^2), \tag{A-13}$$

and this is used in the present work wherever it is required.

In order to complete the normalization and the order-of-magnitude analysis, the remaining characteristic quantities must be specified. For the properties ρ, μ, C_p and k, the properties of the fluid far from the surface might be used as the characteristic quantities, that is:

$$\rho_c = \rho_\infty,$$

$$\mu_c = \mu_\infty,$$

$$C_c = C_{p_\infty}$$

and $$k_c = k_\infty \,.$$

The characteristic length, X_c, is considered to be the position along the surface at which the flow is being examined. This position is sometimes designated as L. The characteristic temperature difference, Θ_c, depends on the surface temperature. For example, if the surface temperature is uniform, then $\Theta_c = \Theta_w$. If the surface temperature is a known function of X, then the largest (or the mean) temperature difference between the surface and the undisturbed fluid might be used for Θ_c. If there is a uniform heat flux along the surface, then Θ_c is defined in terms of the

heat flux, i.e. $\Theta_c = \frac{q_w L}{k}$ where q_w is the heat flux. Having established these remaining characteristic quantities, the order-of-magnitude analysis is complete.

Appendix B - The Falkner-Skan Transformation

The set of equations 2.1-3 is not easily solved even though the order-of-magnitude analysis has already eliminated several terms. In some special cases, it is possible to apply a similarity transformation and obtain a set of ordinary differential equations. The procedure followed in obtaining the similarity transformation can be extended to include a very wide class of problems simply by making the new dependent variables functions of both new independent variables. This is the transformation referred to in Chapter I as the Falkner-Skan transformation and it is applicable to variable-property flows as well as to constant-property flows.

The present work is primarily concerned with constant-property flows and therefore the transformation will be derived on this basis. However, the extension of the procedure for variable-property flows can be easily seen. Starting with the set of equations 2.2-1 and assuming the continuity equation is eliminated by introducing a stream function, the following transformation is applied:

$$\psi(x, y) = x^{N_1} F(\eta, \xi),$$

$$p_d(x, y) = x^{N_2} \Pi(\eta, \xi)$$

and $$\theta(x, y) = x^{N_3} \Phi(\eta, \xi), \qquad \text{(B-1)}$$

where $$\xi = x$$

and $$\eta(x, y) = \frac{y}{C_1} x^{N_4}.$$

The exponents in the above transformation will be functions of

the angle α. The limiting values of these functions can be determined by considering the transformation for $\alpha = 0$ and $\alpha = \Pi/2$. When the transformation is applied to the equations, each term in the equations contains a power of x. Following a procedure used in similarity transformations, the exponents of x in terms not involving derivatives with respect to ξ are equated. If the resulting relations lead to more than one definition of the exponents, then each possible transformation should be considered. If there are no definite advantages of one transformation over the other, then the choice becomes arbitrary. After choosing transformations for $\alpha = 0$ and $\alpha = \Pi/2$, then an attempt is made to establish a general transformation for all angles.

Consider the exponents for $\alpha = 0$. One relation between the exponents is obtained from the conduction and advection terms in the energy equation, and it states that:

$$N_1 + N_3 + N_4 - 1 = N_3 + 2N_4 . \tag{B-2}$$

Another relation, obtained from the pressure and buoyancy terms in the lateral momentum equation is:

$$N_2 + N_4 = N_3 . \tag{B-3}$$

A third relation, obtained from the pressure and viscous terms in the longitudinal momentum equation, is:

$$N_2 - 1 = N_1 + 3N_4 . \tag{B-4}$$

Since a fourth relation is not available, it is assumed that:

$$N_3 = n . \tag{B-5}$$

Solving for the other exponents gives:

$$N_1 = \frac{n+3}{5},$$

$$N_2 = \frac{4n+2}{5} \tag{B-6}$$

and

$$N_4 = \frac{n-2}{5}.$$

Consider the exponents for $\alpha = \Pi/2$. It is assumed that the relation B-2 is also valid for $\alpha = \Pi/2$. Then using the viscous and buoyancy terms in the longitudinal momentum equation, a second relation is:

$$N_1 + 3N_4 = N_3. \tag{B-7}$$

It is also assumed that relations B-4 and B-5 hold for $\alpha = \Pi/2$. The resulting set of exponents is:

$$N_1 = \frac{n+3}{4},$$

$$N_2 = n+1 \tag{B-8}$$

and

$$N_4 = \frac{n-1}{4}$$

A generalization of the exponents given in B-6 and B-8 is given by the following:

$$N_1 = \frac{n+3}{d(\omega)},$$

$$N_2 = 1 + \frac{n+3}{d(\omega)} + \frac{3[n+3-d(\omega)]}{d(\omega)} \qquad \text{(B-9)}$$

and $$N_4 = \frac{n+3-d(\omega)}{d(\omega)},$$

where $d(\omega)$ is any continuous function of ω satisfying

$$d(0) = 5,$$

$$d(1/2) = 4$$

and that $4 \leq d(\omega) \leq 5$.

Using the exponents given by relations B-5 and B-9 in the transformation given by B-1 and introducing the transformation into equations 2.2-1, the following set of equations is obtained:

$$Ra_L^{\frac{1}{5}(4\omega^2-1)}\, x^{\frac{4n+12-3d}{d}} \{(\tfrac{2n+6-d}{d})F_\eta^{\,2} - (\tfrac{n+3}{d})FF_{\eta\eta} + \xi(F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta})$$

$$- AC_1 q\xi^{\frac{d-n-3}{d}} [(\tfrac{n+3}{d})FF_\eta + (\tfrac{n+3-d}{d})\eta F_\eta^{\,2}$$

$$+ \xi F_\xi F_\eta]/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}})\}/[C_1^{\,2}(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}})]$$

$$= - \sigma Ra_L^{-\frac{1}{5}(1+\omega^2)} x^{\frac{4n+12-3d}{d}} [(\tfrac{4n+12-2d}{d})\Pi + (\tfrac{n+3-d}{d})\eta\Pi_\eta$$

$$+ \xi\Pi_\xi]/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}}) + \frac{\sigma}{C_1^{\,3}} Ra_L^{\frac{1}{5}(4\omega^2-1)} x^{\frac{4n+12-3d}{d}} \{F_{\eta\eta\eta}$$

$$+ AC_1 q\xi^{\frac{d-n-3}{d}} F_{\eta\eta}/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}}) - A^2C_1^{\,2}q^2\xi^{\frac{2d-2n-6}{d}} F_\eta/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}})^2\}$$

$$+ \sigma(1+a)x^n\Phi\sin\alpha \tag{B-10}$$

$$- Ra_L^{\omega^2} \frac{Aq}{C_1^{\,2}} x^{\frac{4n+12-2d}{d}} F_\eta^{\,2}/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}}) = - \frac{\sigma}{C_1} x^{\frac{5n+15-3d}{d}} \Pi_\eta + \sigma(1+a)x^n\Phi\cos\alpha$$

$$\frac{1}{C_1} x^{\frac{2n+6-2d+nd}{d}} \{n\Phi F_\eta - (\frac{n+3}{d})\Phi_\eta F + \xi[\Phi_\xi F_\eta - \Phi_\eta F_\xi + \frac{a_\xi}{(1+a)}\Phi F_\eta]\}/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}})$$

$$= \frac{1}{C_1^{\,2}} x^{\frac{2n+6-2d+nd}{d}} \{\Phi_{\eta\eta} + AC_1 q\xi^{\frac{d-n-3}{d}} \Phi_\eta/(1+AC_1 q\eta\xi^{\frac{d-n-3}{d}})\}$$

For the present work, the function $d(\omega)$ has been chosen as

$$d(\omega) = 5 - 4\omega^2.$$

In the transformation B-1, it has been assumed that a stream function can be introduced. For some free-convection flows, e.g. certain axisymmetric flows, it may be impossible or inconvenient to introduce a stream function, in which case the transformation B-1 might be replaced by:

$$u(x, y) = x^{N_1} F_1(\eta, \xi),$$

$$v(x, y) = x^{N_2} F_2(\eta, \xi),$$

$$p_d(x, y) = x^{N_3} \Pi(\eta, \xi),$$

and

$$\theta(x, y) = x^{N_4} \Phi(\eta, \xi)$$

where

$$\xi = x$$

and

$$\eta(x, y) = \frac{y}{C_1} x^{N_5}.$$

Appendix C - The Lefevre Transformation

The set of equations 2.2-6 is applicable to fluids having Prandtl numbers of unit order or higher. An attempt will now be made to transform the equations such that the new equations are applicable for all Prandtl numbers, including zero and infinity.

Consider the following transformation:

$$\eta = \Xi_1(\sigma)\zeta,$$

$$F(\eta,\ \xi) = \Xi_2(\sigma)f(\zeta,\ \xi),$$

$$\Pi(\eta,\ \xi) = \Xi_3(\sigma)\pi(\zeta,\ \xi)$$

and $$\Phi(\eta,\ \xi) = \chi(\zeta,\ \xi),$$

where $\Xi_1(\sigma)$, $\Xi_2(\sigma)$ and $\Xi_3(\sigma)$ are disposable functions of the Prandtl number. Introducing this transformation into the set of equations 2.2-6 leads to the following set of equations:

$$C_1 Ra_L^{\frac{1}{5}(4\omega^2-1)}\{(\frac{2n+1+4\omega^2}{5-4\omega^2})\frac{\Xi_2{}^2}{\Xi_1{}^2}f_\zeta{}^2 - (\frac{n+3}{5-4\omega^2})\frac{\Xi_2{}^2}{\Xi_1{}^2}ff_{\zeta\zeta} + \frac{\xi\Xi_2{}^2}{\Xi_1{}^2}(f_\zeta f_{\xi\zeta} - f_\xi f_{\zeta\zeta})$$

$$- C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}\frac{\Xi_2{}^2}{\Xi_1}[(\frac{n+3}{5-4\omega^2})ff_\zeta + (\frac{n-2+4\omega^2}{5-4\omega^2})\zeta f_\zeta{}^2$$

$$+ \xi f_\xi f_\zeta]/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})\}/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

$$= -\sigma C_1{}^3 Ra_L{}^{-\frac{1}{5}(1+\omega^2)} \Xi_3[(\frac{4n+2+8\omega^2}{5-4\omega^2})\pi + (\frac{n-2+4\omega^2}{5-4\omega^2})\zeta\pi_\zeta$$

$$+ \xi\pi_\xi]/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}) + \sigma Ra_L{}^{\frac{1}{5}(4\omega^2-1)} \{\frac{\Xi_2}{\Xi_1{}^3} f_{\zeta\zeta\zeta}$$

$$+ C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} \frac{\Xi_2}{\Xi_1{}^2} f_{\zeta\zeta}/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

$$- C_1{}^2 A^2 q^2 \xi^{\frac{4-2n-8\omega^2}{5-4\omega^2}} \frac{\Xi_2}{\Xi_1} f_\zeta/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})^2\}$$

$$+ \sigma(1+a)C_1{}^3\xi^{\frac{(n+3)(1-4\omega^2)}{5-4\omega^2}} \chi\sin\alpha \qquad \text{(C-1)}$$

$$- Ra_L{}^{\omega^2} Aq\xi^{\frac{(2-n)(1-4\omega^2)}{5-4\omega^2}} \frac{\Xi_2{}^2 f_\zeta{}^2}{\Xi_1{}^2}/[C_1{}^2(1+Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})]$$

$$= \frac{-\sigma\xi^{\frac{4\omega^2(n+3)}{5-4\omega^2}}}{C_1} \frac{\Xi_3}{\Xi_1}\pi_\zeta + \sigma(1+a)\chi\cos\alpha$$

$$C_1\{\frac{n\Xi_2}{\Xi_1}\chi f_\zeta - (\frac{n+3}{5-4\omega^2})\frac{\Xi_2}{\Xi_1}\chi_\zeta f + \frac{\xi\Xi_2}{\Xi_1}[\chi_\xi f_\zeta - \chi_\zeta f_\xi + \frac{\chi f_\zeta}{(1+a)} a_\xi]\}/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

$$= \frac{1}{\Xi_1{}^2}\chi_{\zeta\zeta} + \frac{C_1 Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}}{\Xi_1} \chi_\zeta/(1+C_1 Aq\Xi_1\zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}})$$

For the energy equation in C-1, since the flow is thermally generated, it is assumed that the advection terms and one of the conduction terms will be of the same order for the limits of $\sigma = 0$ and $\sigma \to \infty$, implying that

$$\Xi_2 \sim \frac{1}{\Xi_1}. \qquad \text{(C-2)}$$

In the lateral momentum equation, it is also assumed that the pressure term and the buoyancy term will be of the same order for the limits of $\sigma = 0$ and $\sigma \to \infty$, implying that

$$\frac{\Xi_3}{\Xi_1} \sim 1. \qquad \text{(C-3)}$$

A third relationship can be obtained from the longitudinal momentum equation, but this relationship depends on α. For $\alpha = 0$, it is assumed that for small Prandtl numbers the order of the pressure term is the same as that of the inertia terms, such that

$$\sigma\Xi_3 \sim \frac{\Xi_2^{\,2}}{\Xi_1^{\,2}}. \qquad \text{(C-4)}$$

For $\alpha = 0$ and for large Prandtl numbers, the order of the pressure term is assumed to be equal to that of one of the viscous terms, implying that

$$\Xi_3 \sim \frac{\Xi_2}{\Xi_1^{\,3}}. \qquad \text{(C-5)}$$

For the present work, the above relations will be taken as identities. Then a combination of C-2, C-3 and C-4 implies that

$$\Xi_3^{\,5} = \frac{1}{\sigma}, \qquad \text{(C-6)}$$

whereas a combination of C-2, C-3 and C-5 implies that

$$\Xi_3{}^5 = 1 \tag{C-7}$$

A generalization of C-6 and C-7 is given by

$$\Xi_3 = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5}} . \tag{C-8}$$

Then Ξ_1 and Ξ_2 are given by

$$\Xi_1 = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{5}}$$

and

$$\Xi_2 = \left(\frac{1+\sigma}{\sigma}\right)^{-\frac{1}{5}} . \tag{C-9}$$

For $\alpha = \Pi/2$, it is assumed that the buoyancy term and the inertia terms have the same order for small Prandtl numbers, implying that

$$\frac{\Xi_2{}^2}{\Xi_1{}^2} \sim \sigma . \tag{C-10}$$

For large Prandtl numbers, it is assumed that order of the buoyancy term is the same as that of one of the viscous terms, implying that

$$\frac{\Xi_2}{\Xi_1{}^3} \sim 1 . \tag{C-11}$$

For the present work, C-10 and C-11 will be assumed to be identities. Then combining C-2 and C-10 leads to

$$\Xi_1{}^4 = \frac{1}{\sigma} , \tag{C-12}$$

and combining C-2 and C-11 leads to

$$\Xi_1^{\,4} = 1. \tag{C-13}$$

A generalization similar to that used in C-8 results in the following expressions for Ξ_1, Ξ_2 and Ξ_3:

$$\Xi_1 = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{4}},$$

$$\Xi_2 = \left(\frac{1+\sigma}{\sigma}\right)^{-\frac{1}{4}} \tag{C-14}$$

and
$$\Xi_3 = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{1}{4}}.$$

The expressions for Ξ_1, Ξ_2 and Ξ_3 given in C-8, C-9 and C-14 may be generalized as follows:

$$\Xi_1 = \left(\frac{1+\sigma}{\sigma}\right)^{h},$$

$$\Xi_2 = \left(\frac{1+\sigma}{\sigma}\right)^{-h} \tag{C-15}$$

and
$$\Xi_3 = \left(\frac{1+\sigma}{\sigma}\right)^{h},$$

where $h = h(\omega)$ is any continuous function of ω satisfying the conditions that

$$h(0) = \frac{1}{5},$$

$$h\left(\frac{1}{2}\right) = \frac{1}{4}$$

and that $\frac{1}{5} \leq h(\omega) \leq \frac{1}{4}$.

Substituting C-15 into C-1 leads to the following set of equations:

$$C_1 Ra_L^{\frac{1}{5}(4\omega^2-1)} \{ (\frac{2n+1+4\omega^2}{5-4\omega^2}) f_\zeta^{\ 2} - (\frac{n+3}{5-4\omega^2}) f f_{\zeta\zeta} + \xi (f_\zeta f_{\xi\zeta} - f_\xi f_{\zeta\zeta})$$

$$- C_1 Aq \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} (\frac{1+\sigma}{\sigma})^h [(\frac{n+3}{5-4\omega^2}) f f_\zeta + (\frac{n-2+4\omega^2}{5-4\omega^2}) \zeta f_\zeta^{\ 2}$$

$$+ \xi f_\xi f_\zeta] / [1 + C_1 Aq (\frac{1+\sigma}{\sigma})^h \zeta \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}] \} / [1 + C_1 Aq (\frac{1+\sigma}{\sigma})^h \zeta \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]$$

$$= - \sigma (\frac{1+\sigma}{\sigma})^{5h} C_1^{\ 3} Ra_L^{-\frac{1}{5}(1+\omega^2)} [(\frac{4n+2+8\omega^2}{5-4\omega^2}) \pi + (\frac{n-2+4\omega^2}{5-4\omega^2}) \zeta \pi_\zeta$$

$$+ \xi \pi_\xi] / [1 + C_1 Aq (\frac{1+\sigma}{\sigma})^h \zeta \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}] + \sigma Ra_L^{\frac{1}{5}(4\omega^2-1)} \{ f_{\zeta\zeta\zeta}$$

$$+ C_1 Aq \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} (\frac{1+\sigma}{\sigma})^h f_{\zeta\zeta} / [1 + C_1 Aq (\frac{1+\sigma}{\sigma})^h \zeta \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]$$

$$- C_1^{\ 2} A^2 q^2 \xi^{\frac{4-2n-8\omega^2}{5-4\omega^2}} (\frac{1+\sigma}{\sigma})^{2h} f_\zeta / [1 + C_1 Aq (\frac{1+\sigma}{\sigma})^h \zeta \xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]^2 \}$$

$$+ \sigma(1+a) (\frac{1+\sigma}{\sigma})^{4h} C_1^{\ 3} \xi^{\frac{(n+3)(1-4\omega^2)}{5-4\omega^2}} \chi \sin\alpha \qquad \text{(C-16)}$$

$$- q\xi^{\frac{(2-n)(1-4\omega^2)}{5-4\omega^2}} \left(\frac{1+\sigma}{\sigma}\right)^{-4h} f_\zeta{}^2/\{C_1{}^2[1+C_1Aq\left(\frac{1+\sigma}{\sigma}\right)^h \zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]\}$$

$$= \frac{-\sigma\xi}{C_1}^{\frac{4\omega^2(n+3)}{5-4\omega^2}} \pi_\zeta + \sigma(1+a)\chi\cos\alpha$$

$$C_1\{n\chi f_\zeta - \left(\frac{n+3}{5-4\omega^2}\right)\chi_\zeta f + \xi[\chi_\xi f_\zeta - \chi_\zeta f_\xi + \frac{\chi f\zeta a_\xi}{(1+a)}]\}/[1+C_1Aq\left(\frac{1+\sigma}{\sigma}\right)^h \zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]$$

$$= \chi_{\zeta\zeta} + C_1Aq\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}} \left(\frac{1+\sigma}{\sigma}\right)^h \chi_\zeta/[1+C_1Aq\left(\frac{1+\sigma}{\sigma}\right)^h \zeta\xi^{\frac{2-n-4\omega^2}{5-4\omega^2}}]$$

For the present work, the function $h(\omega)$ has been chosen as

$$h(\omega) = \frac{1}{5-4\omega^2} .$$

This procedure can also be applied to variable-property flows.

Appendix D - A Normalization and an Order-of-Magnitude Analysis of the Disturbance Equations

Following the procedure outlined in appendix A for the mean-flow equations, an attempt is made to normalize the dependent and independent variables in the set of equations 2.3-1 by introducing a set of characteristic quantities. The definitions of these quantities must follow either from the boundary conditions, from the equations or from the physical description of the problem. Since the disturbance quantities are assumed to be small, it follows that:

$$X_c{}^* << \bar{X}_c$$

$$Y_c{}^* << \bar{X}_c$$

$$Z_c{}^* << \bar{X}_c$$

$$U_c{}^* << \bar{U}_c$$

$$V_c{}^* << \bar{U}_c$$

$$W_c{}^* << \bar{U}_c$$

$$\Theta_c{}^* << \bar{\Theta}_c$$

$$\rho_c{}^* << \bar{\rho}_c$$

$$\mu_c{}^* << \bar{\mu}_c$$

$$C_c^* << \bar{C}_c$$

$$k_c^* << \bar{k}_c$$

$$P_c^* << \bar{P}_c$$

The analysis of the mean-flow equations reveals that

$$\bar{Y}_c << \bar{X}_c \text{ and } \bar{V}_c << \bar{U}_c.$$

These last relationships suggest that the characteristic lengths and velocities for the disturbance flow might be related to the characteristic lengths and velocities for the mean flow as follows:

$$X_c^* = Y_c^* = Z_c^* = \bar{Y}_c = \bar{X}_c \; Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

and

$$U_c^* = V_c^* = W_c^* = \bar{V}_c = \frac{\bar{\nu}_c}{\sigma \bar{X}_c} \; Ra_{\bar{X}_c}^{\frac{1}{2}H}$$

The characteristic time, t_c, is chosen in such a way that the highest order terms involving derivatives with respect to time are of the same order as the highest order terms retained in each equation. The characteristic time and the remaining relationships between the characteristic quantities for the disturbance flow and the characteristic quantities for the mean flow may be determined by assuming that:

$$t_c = \frac{X_c^*}{\bar{U}_c} = \frac{\sigma \bar{X}_c^{\,2}}{\bar{\nu}_c} \; Ra_{\bar{X}_c}^{-\frac{3}{2}H}$$

$$\frac{\rho_c^*}{\bar{\rho}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

$$\frac{\Theta_c^*}{\bar{\Theta}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

$$\frac{\mu_c^*}{\bar{\mu}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

$$\frac{C_c^*}{\bar{C}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

$$\frac{P_c^*}{\bar{P}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}$$

$$\frac{k_c^*}{\bar{k}_c} = \frac{X_c^*}{\bar{X}_c} = Ra_{\bar{X}_c}^{-\frac{1}{2}H}.$$

Following the approach used for the mean-flow equations, $\bar{X}_c$ is replaced by L. Terms of the order of Ra_L^{-H}, σRa_L^{-H}, JRa_L^{-H}, $A^2\sigma Ra_L^{-H}$, $J^2\sigma Ra_L^{-H}$, $A^2 J\sigma Ra_L^{-H}$, $[\frac{T}{\rho}(\frac{\partial\rho}{\partial T})_P]$ $OsRa_L^{-\frac{1}{2}H}$, $OsRa_L^{\frac{3}{2}H-1}$, $A^2 OsRa_L^{\frac{3}{2}H-1}$, $J^2 OsRa_L^{H-1}$, $AJOsRa_L^{H-1}$ or less relative to other terms in the equations are assumed to be negligible. Therefore, after eliminating the pressure terms from the disturbance momentum equations, normalizing the equations, and neglecting terms of the orders indicated above, a simplified set of

disturbance equations is obtained as follows:

$$\frac{\partial \gamma_d^*}{\partial \tau} + (1+\bar{\gamma}_d)[\frac{\partial v^*}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\partial u^*}{\partial x^*} + Aqv^*) + \frac{1}{(j\pm Jysin\alpha)}(\frac{\partial w}{\partial z} \pm v^*Jsin\alpha)] + \frac{v^*\partial \bar{\gamma}_d}{\partial y}$$

$$+ \frac{\bar{u}}{(1+Aqy)}\frac{\partial \gamma_d^*}{\partial x^*} + Ra_L^{-\frac{1}{2}H}\{\frac{1}{(j\pm Jysin\alpha)}(\frac{dj}{d\bar{x}} \pm Jycos\alpha\frac{d\alpha}{d\bar{x}})[\frac{(1+\bar{\gamma}_d)u^*}{(1+Aqy)}$$

$$+ \frac{\gamma_d^*\bar{u}}{(1+Aqy)}] \pm \frac{\gamma_d^*\bar{v}Jsin\alpha}{(j\pm Jysin\alpha)} + \frac{1}{(1+Aqy)}[\gamma_d^*(\frac{\partial \bar{u}}{\partial \bar{x}} + Aq\bar{v}) + \frac{u^*\partial \bar{\gamma}_d}{\partial \bar{x}}]$$

$$+ \gamma_d^*\frac{\partial \bar{v}}{\partial y} + \frac{\bar{v}\partial \gamma_d^*}{\partial y}\} = 0$$

$$(1+\bar{\gamma}_d)\{\frac{\partial^2 w}{\partial y\partial \tau} + \frac{1}{(1+Aqy)}[\frac{\bar{u}\partial^2 w}{\partial y\partial x^*} + \frac{\partial \bar{u}}{\partial y}\frac{\partial w}{\partial x^*} - \frac{Aq\bar{u}}{(1+Aqy)}\frac{\partial w}{\partial x^*}]$$

$$+ \frac{1}{(j\pm Jysin\alpha)}[\frac{1}{(1+Aqy)}(\frac{2Aq\bar{u}\partial u^*}{\partial z} \pm \frac{\bar{u}Jsin\alpha\partial w}{\partial x^*} - \frac{\bar{u}\partial^2 v^*}{\partial z\partial x^*}) \pm \frac{Jsin\alpha\partial w}{\partial \tau}$$

$$- \frac{\partial^2 v^*}{\partial z\partial \tau}]\} + \frac{\partial \bar{\gamma}_d}{\partial y}\frac{\partial w}{\partial \tau} + \frac{\bar{u}}{(1+Aqy)}\frac{\partial \bar{\gamma}_d}{\partial y}\frac{\partial w}{\partial x^*} + \frac{Aq\bar{u}^2}{(1+Aqy)(j\pm Jysin\alpha)}\frac{\partial \gamma_d^*}{\partial z}$$

$$+ Ra_L^{-\frac{1}{2}H}\{(1+\bar{\gamma}_d)(\frac{\bar{v}\partial^2 w}{\partial y^2} + \frac{\partial \bar{v}}{\partial y}\frac{\partial w}{\partial y}) + \frac{(1+\bar{\gamma}_d)}{(j\pm Jysin\alpha)}[\frac{1}{(1+Aqy)}(\frac{dj}{d\bar{x}}$$

$$\pm Jycos\alpha\frac{d\alpha}{d\bar{x}})(\frac{w\partial \bar{u}}{\partial y} + \frac{\bar{u}\partial w}{\partial y} - \frac{Aqw\bar{u}}{(1+Aqy)}) - \frac{\bar{v}\partial^2 v^*}{\partial z\partial y} \pm \frac{2\bar{v}Jsin\alpha\partial w}{\partial y}$$

$$\pm \frac{w\bar{u}J\cos\alpha}{(1+Aqy)} \frac{d\alpha}{d\bar{x}} - \frac{\partial\bar{v}}{\partial y} \frac{\partial v^*}{\partial z} \pm \frac{wJ\sin\alpha\partial\bar{v}}{\partial y}] + \frac{\bar{v}\partial\bar{\gamma}_d}{\partial y} \frac{\partial w}{\partial y}\}$$

$$= \sigma Ra_L^{-\frac{1}{2}H}\{\frac{1}{(j\pm Jy\sin\alpha)} \frac{\partial^2 M^*}{\partial z\partial x^*}[\frac{Aq\bar{u}}{(1+Aqy)^2} - \frac{1}{(1+Aqy)} \frac{\partial\bar{u}}{\partial y}] + \frac{\partial^2\bar{M}}{\partial y^2}[\frac{\partial w}{\partial y}$$

$$+ \frac{1}{(j\pm Jy\sin\alpha)}(\frac{\partial v^*}{\partial z} \mp wJ\sin\alpha)] + \frac{\partial\bar{M}}{\partial y}[\frac{2\partial^2 w}{\partial y^2} + \frac{Aq}{(1+Aqy)} \frac{\partial w}{\partial y}$$

$$+ \frac{1}{(1+Aqy)^2} \frac{\partial^2 w}{\partial x^{*2}} \mp \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)} \frac{\partial w}{\partial y} + \frac{1}{(1+Aqy)(j\pm Jy\sin\alpha)}(\frac{\partial^2 u^*}{\partial z\partial x^*} + \frac{Aq\partial v^*}{\partial z}$$

$$\pm AqwJ\sin\alpha) + \frac{1}{(j\pm Jy\sin\alpha)^2}(\frac{2\partial^2 w}{\partial z^2} \pm \frac{3J\sin\alpha\partial v^*}{\partial z} - wJ^2\sin^2\alpha)]$$

$$+ \bar{M}[\frac{\partial^3 w}{\partial y^3} + \frac{Aq}{(1+Aqy)} \frac{\partial^2 w}{\partial y^2} + \frac{1}{(1+Aqy)^2}(\frac{\partial^3 w}{\partial y\partial x^{*2}} + \frac{2Aq\partial^2 u^*}{\partial z\partial x^*} - \frac{A^2q^2\partial w}{\partial y}$$

$$- \frac{\partial^3 v^*}{\partial z\partial x^{*2}}) - \frac{2Aq}{(1+Aqy)^3} \frac{\partial^2 w}{\partial x^{*2}}] + \frac{\bar{M}}{(j\pm Jy\sin\alpha)}[\frac{\partial^3 w}{\partial z^2\partial y} \pm \frac{2J\sin\alpha\partial^2 w}{\partial y^2} - \frac{\partial^3 v^*}{\partial y^2\partial z}$$

$$+ \frac{1}{(1+Aqy)}(\pm \frac{2AqJ\sin\alpha\partial w}{\partial y} - \frac{Aq\partial^2 v^*}{\partial y\partial z}) + \frac{1}{(1+Aqy)^2}(\frac{A^2q^2\partial v^*}{\partial z}$$

$$\pm \frac{J\sin\alpha\partial^2 w}{\partial x^{*2}} \mp A^2q^2wJ\sin\alpha)] + \frac{\bar{M}}{(j\pm Jy\sin\alpha)^2}(\pm \frac{J\sin\alpha\partial^2 v^*}{\partial y\partial z}$$

$$- \frac{J^2\sin^2\alpha\partial w}{\partial y} \pm \frac{J\sin\alpha\partial^2 w}{\partial z^2} + \frac{\partial^3 v^*}{\partial z^3}) + \frac{\bar{M}J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^3}(\pm wJ\sin\alpha$$

$$- \frac{\partial v^*}{\partial z})\} + \frac{\sigma\cos\alpha}{\beta\bar{\Theta}_c} Ra_L^{1-\frac{5}{2}H} \frac{\partial \gamma_d^*}{\partial z}$$

$$(1+\bar{\gamma}_d)\{\frac{1}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 u^*}{\partial z \partial \tau} + \frac{\partial \bar{u}}{\partial y}\frac{\partial v^*}{\partial z} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2 u^*}{\partial z \partial x^*} + \frac{Aq\bar{u}\partial v^*}{\partial z})] - \frac{\bar{u}}{(1+Aqy)^2}\frac{\partial^2 w}{\partial x^{*2}}$$

$$- \frac{1}{(1+Aqy)}\frac{\partial^2 w}{\partial x^* \partial \tau}\} + Ra_L^{-\frac{1}{2}H}\{\frac{(1+\bar{\gamma}_d)}{(1+Aqy)}[\frac{Ay\bar{u}}{(1+Aqy)^2}\frac{dq}{d\bar{x}}\frac{\partial w}{\partial x^*}$$

$$- \frac{1}{(1+Aqy)}\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial w}{\partial x^*} - \frac{\bar{v}\,\partial^2 w}{\partial x^* \partial y}] + \frac{(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}[\frac{\bar{v}\partial^2 u^*}{\partial z \partial y} + \frac{1}{(1+Aqy)}(\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u^*}{\partial z}$$

$$+ \frac{Aq\bar{v}\partial u^*}{\partial z} \mp \frac{\bar{v}J\sin\alpha\partial w}{\partial x^*})] - \frac{(1+\bar{\gamma}_d)}{(j\pm Jy\sin\alpha)}(\frac{dj}{d\bar{x}} \pm \frac{Jy\cos\alpha d\alpha}{d\bar{x}})[\frac{1}{(1+Aqy)}\frac{\partial w}{\partial \tau}$$

$$+ \frac{2\bar{u}}{(1+Aqy)^2}\frac{\partial w}{\partial x^*}] - \frac{1}{(1+Aqy)}\frac{\partial \bar{\gamma}_d}{\partial \bar{x}}[\frac{\partial w}{\partial \tau} + \frac{\bar{u}}{(1+Aqy)}\frac{\partial w}{\partial x^*}]$$

$$+ \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial \gamma_d^*}{\partial z}[\frac{\bar{v}\partial \bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial \bar{u}}{\partial \bar{x}} + Aq\bar{v})]\}$$

$$= \sigma Ra_L^{-\frac{1}{2}H}\{\frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial^2 M^*}{\partial z \partial y}[\frac{\partial \bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}] + \frac{\partial \bar{M}}{\partial y}[\frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial^2 u^*}{\partial z \partial y}$$

$$- \frac{1}{(1+Aqy)}\frac{\partial^2 w}{\partial x^* \partial y}] - \frac{1}{(j\pm Jy\sin\alpha)}\frac{\partial \bar{M}}{\partial y}[\frac{Aq}{(1+Aqy)}\frac{\partial u^*}{\partial z} \mp \frac{J\sin\alpha}{(1+Aqy)}\frac{\partial w}{\partial x^*}]$$

$$- \bar{M}[\frac{1}{(1+Aqy)^3}\frac{\partial^3 w}{\partial x^{*3}} + \frac{Aq}{(1+Aqy)^2}\frac{\partial^2 w}{\partial x^* \partial y} + \frac{1}{(1+Aqy)}\frac{\partial^3 w}{\partial x^* \partial y^2}]$$

$$+ \frac{\bar{M}}{(j \pm Jy\sin\alpha)}[\frac{\partial^3 u^*}{\partial z \partial y^2} + \frac{1}{(1+Aqy)}(Aq\frac{\partial^2 u^*}{\partial z \partial y} \mp J\sin\alpha \frac{\partial^2 w}{\partial x^* \partial y})$$

$$+ \frac{1}{(1+Aqy)^2}(2Aq\frac{\partial^2 v^*}{\partial z \partial x^*} - A^2 q^2 \frac{\partial u^*}{\partial z} + \frac{\partial^3 u^*}{\partial x^{*2} \partial z} \mp AwJ\sin\alpha\frac{dq}{d\bar{x}} \mp AqJ\sin\alpha\frac{\partial w}{\partial x^*})]$$

$$+ \frac{\bar{M}}{(j \pm Jy\sin\alpha)^2}[\pm J\sin\alpha\frac{\partial^2 u^*}{\partial z \partial y} + \frac{1}{(1+Aqy)}(J^2\sin^2\alpha\frac{\partial w}{\partial x^*} \mp 2J\sin\alpha\frac{\partial^2 v^*}{\partial z \partial x^*}$$

$$\pm AqJ\sin\alpha\frac{\partial u^*}{\partial z} - \frac{\partial^3 w}{\partial z^2 \partial x^*})] + \frac{\bar{M}}{(j \pm Jy\sin\alpha)^3}\frac{\partial^3 u^*}{\partial z^3} + \frac{1}{(j \pm Jy\sin\alpha)}\frac{\partial M^*}{\partial z}[\frac{\partial^2 \bar{u}}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial \bar{u}}{\partial y} - \frac{A^2 q^2 \bar{u}}{(1+Aqy)^2}] \pm \frac{J\sin\alpha}{(j \pm Jy\sin\alpha)^2}\frac{\partial M^*}{\partial z}[\frac{\partial \bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\}$$

$$- \frac{\sigma\sin\alpha}{\beta\bar{\Theta}_c} Ra_L^{1-\frac{5}{2}H}\frac{\partial \gamma_d^*}{\partial z} \qquad \text{(D-1)}$$

$$(1+\bar{\gamma}_d)[\frac{1}{(1+Aqy)}(\frac{\partial^2 v^*}{\partial x^* \partial \tau} - Aq\frac{\partial u^*}{\partial \tau} - 2Aqv^*\frac{\partial \bar{u}}{\partial y} - Aq\bar{u}\frac{\partial v^*}{\partial y} - \frac{\partial \bar{u}}{\partial y}\frac{\partial u^*}{\partial x^*} - \frac{\bar{u}\partial^2 u^*}{\partial y \partial x^*}) - \frac{\partial^2 u^*}{\partial y \partial \tau}$$

$$- v^*\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial v^*}{\partial y}\frac{\partial \bar{u}}{\partial y} - \frac{1}{(1+Aqy)^2}(2Aq\bar{u}\frac{\partial u^*}{\partial x^*} - \frac{\bar{u}\partial^2 v^*}{\partial x^{*2}})] - \frac{\partial \bar{\gamma}_d}{\partial y}[\frac{\partial u^*}{\partial \tau} + v^*\frac{\partial \bar{u}}{\partial y}$$

$$+ \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial u^*}{\partial x^*} + Aq\bar{u}v^*)] - \frac{Aq\bar{u}^2}{(1+Aqy)^2}\frac{\partial \gamma_d^*}{\partial x^*} + Ra_L^{-\frac{1}{2}H}\{\frac{1}{(1+Aqy)}\frac{\partial \bar{\gamma}_d}{\partial \bar{x}}[\frac{\partial v^*}{\partial \tau}$$

$$+ \frac{\bar{u}}{(1+Aqy)}(\frac{\partial v^*}{\partial x^*} - 2Aqu^*)] - (1+\bar{\gamma}_d)[\frac{\bar{v}\partial^2 u^*}{\partial y^2} + \frac{\partial \bar{v}}{\partial y}\frac{\partial u^*}{\partial y} + \frac{1}{(1+Aqy)}(2Aqv\frac{\partial u^*}{\partial y}$$

$$- \frac{\bar{v}\,\partial^2 v^*}{\partial x^*\partial y} - \frac{\partial v^*}{\partial x^*}\frac{\partial \bar{v}}{\partial y} + \frac{u^*\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial u^*}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aqu^*\partial\bar{v}}{\partial y}) + \frac{1}{(1+Aqy)^2}(\frac{2A\bar{u}u^*dq}{d\bar{x}}$$

$$+ \frac{2Aqu^*\partial\bar{u}}{\partial\bar{x}} - \frac{\partial v^*}{\partial x^*}\frac{\partial\bar{u}}{\partial\bar{x}}) + \frac{Ay\bar{u}}{(1+Aqy)^3}\frac{dq}{d\bar{x}}(\frac{\partial v^*}{\partial x^*} - 2Aqu^*)] - \gamma_d{}^*[\frac{\bar{v}\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{1}{(1+Aqy)}(\frac{\bar{u}\partial^2\bar{u}}{\partial y\partial\bar{x}} + \frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{Aq\bar{u}\partial\bar{v}}{\partial y} + \frac{2Aq\bar{v}\partial\bar{u}}{\partial y}) + \frac{A\bar{u}}{(1+Aqy)^2}(\frac{\bar{u}dq}{d\bar{x}}$$

$$+ \frac{2q\partial\bar{u}}{\partial\bar{x}}) - \frac{A^2qy\bar{u}^2}{(1+Aqy)^3}\frac{dq}{d\bar{x}}] - \frac{\partial\bar{\gamma}_d}{\partial y}[\frac{\bar{v}\partial u^*}{\partial y} + \frac{u^*}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]$$

$$- \frac{\partial\gamma_d{}^*}{\partial y}[\frac{\bar{v}\partial\bar{u}}{\partial y} + \frac{\bar{u}}{(1+Aqy)}(\frac{\partial\bar{u}}{\partial\bar{x}} + Aq\bar{v})]\}$$

$$= \sigma Ra_L^{-\frac{1}{2}H}\{\frac{\partial^2\bar{M}}{\partial y^2}[\frac{1}{(1+Aqy)}(Aqu^* - \frac{\partial v^*}{\partial x^*}) - \frac{\partial u^*}{\partial y}] - \frac{\partial\bar{M}}{\partial y}[\frac{2\partial^2u^*}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial u^*}{\partial y}$$

$$+ \frac{1}{(1+Aqy)^2}(\frac{2\partial^2u^*}{\partial x^{*2}} + \frac{3Aq\partial v^*}{\partial x^*} - A^2q^2u^*)] - \frac{1}{(j\pm Jysin\alpha)}\frac{\partial\bar{M}}{\partial y}[\frac{1}{(1+Aqy)}(\frac{\partial^2 w}{\partial x^*\partial z}$$

$$\pm \frac{Jsin\alpha\partial v^*}{\partial x^*} \pm Aqu^*Jsin\alpha) \pm \frac{Jsin\alpha\partial u^*}{\partial y} + \frac{1}{(j\pm Jysin\alpha)}\frac{\partial^2u^*}{\partial z^2}]$$

$$+ \frac{1}{(1+Aqy)^2}\frac{\partial^2M^*}{\partial x^{*2}}[\frac{\partial\bar{u}}{\partial y} - \frac{Aq\bar{u}}{(1+Aqy)}] + \frac{\partial^2M^*}{\partial y^2}[\frac{Aq\bar{u}}{(1+Aqy)} - \frac{\partial\bar{u}}{\partial y}] - \frac{\partial M^*}{\partial y}[\frac{2\partial^2\bar{u}}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial\bar{u}}{\partial y} - \frac{A^2q\bar{u}}{(1+Aqy)^2}] \mp \frac{Jsin\alpha}{(j\pm Jysin\alpha)}\frac{\partial M^*}{\partial y}[\frac{\partial\bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]$$

$$- \bar{M}[\frac{\partial^3 u^*}{\partial y^3} + \frac{1}{(1+Aqy)}(2Aq\frac{\partial^2 u^*}{\partial y^2} - \frac{\partial^3 v^*}{\partial y^2 \partial x^*}) + \frac{1}{(1+Aqy)^2}(\frac{\partial^3 u^*}{\partial x^{*2}\partial y} + Aq\frac{\partial^2 v^*}{\partial y \partial x^*}$$

$$- A^2q^2\frac{\partial u^*}{\partial y}) - \frac{1}{(1+Aqy)^3}(\frac{\partial^3 v^*}{\partial x^{*3}} - Aq\frac{\partial^2 u^*}{\partial x^{*2}} + A^2q^2\frac{\partial v^*}{\partial x^*} - A^3q^3u^*)]$$

$$\mp \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 u^*}{\partial y^2} + \frac{1}{(1+Aqy)}(2Aq\frac{\partial u^*}{\partial y} - \frac{\partial^2 v^*}{\partial y \partial x^*})]$$

$$+ \frac{\bar{M}}{(j\pm Jy\sin\alpha)^2}[J^2\sin^2\alpha\frac{\partial u^*}{\partial y} - \frac{\partial^3 u^*}{\partial y \partial z^2} - \frac{1}{(1+Aqy)}(\pm J\sin\alpha\frac{\partial^2 w}{\partial x^* \partial z} - \frac{\partial^3 v^*}{\partial x^* \partial z^2}$$

$$+ J^2\sin^2\alpha\frac{\partial v^*}{\partial x^*} - Aqu^*J^2\sin^2\alpha + Aq\frac{\partial^2 u^*}{\partial z^2} \pm J\sin\alpha\frac{\partial^2 w}{\partial z \partial x^*})]$$

$$\pm \frac{\bar{M}J\sin\alpha}{(j\pm Jy\sin\alpha)^3}\frac{\partial^2 u^*}{\partial z^2} - M^*[\frac{\partial^3 \bar{u}}{\partial y^3} + \frac{2Aq}{(1+Aqy)}\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{A^2q^2}{(1+Aqy)^2}\frac{\partial \bar{u}}{\partial y} + \frac{A^3q^3\bar{u}}{(1+Aqy)^3}]$$

$$\mp \frac{M^*J\sin\alpha}{(j\pm Jy\sin\alpha)}[\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{2Aq}{(1+Aqy)}\frac{\partial \bar{u}}{\partial y}] + \frac{M^*J^2\sin^2\alpha}{(j\pm Jy\sin\alpha)^2}[\frac{\partial \bar{u}}{\partial y} + \frac{Aq\bar{u}}{(1+Aqy)}]\}$$

$$+ \frac{\sigma Ra_L}{\beta\bar{\Theta}_c}^{1-\frac{5}{2}H}[\frac{\partial \gamma_d^*\sin\alpha}{\partial y} + \frac{\gamma_d^*Aq\sin\alpha}{(1+Aqy)} - \frac{\cos\alpha}{(1+Aqy)}\frac{\partial \gamma_d^*}{\partial x^*} + Ra_L^{-\frac{1}{2}H}\frac{\gamma_d^*\sin\alpha}{(1+Aqy)}\frac{d\alpha}{d\bar{x}}]$$

$$(1+\bar{\gamma}_d)\bar{c}(1+a)[\frac{\partial \theta^*}{\partial \tau} + \frac{\bar{u}}{(1+Aqy)}\frac{\partial \theta^*}{\partial x^*} + v^*\frac{\partial \bar{\theta}}{\partial y}] + Ra_L^{-\frac{1}{2}H}\{(1+\bar{\gamma}_d)\bar{c}[\frac{1}{(1+Aqy)}\frac{da}{d\bar{x}}(\bar{u}\theta^*$$

$$+ u^*\bar{\theta}) + \bar{v}(1+a)\frac{\partial \theta^*}{\partial y} + \frac{u^*(1+a)}{(1+Aqy)}\frac{\partial \bar{\theta}}{\partial \bar{x}}] + \gamma_d^*\bar{c}[\frac{\bar{u}(1+a)}{(1+Aqy)}\frac{\partial \bar{\theta}}{\partial \bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}}$$

$$+ \bar{v}(1+a)\frac{\partial\bar{\theta}}{\partial y}] + (1+\bar{\gamma}_d)c^*[\frac{\bar{u}(1+a)}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial\bar{x}} + \frac{\bar{u}\bar{\theta}}{(1+Aqy)}\frac{da}{d\bar{x}} + \bar{v}(1+a)\frac{\partial\bar{\theta}}{\partial y}]\}$$

$$= (1+a)Ra_L^{-\frac{1}{2}}H\{K^*[\frac{\partial^2\bar{\theta}}{\partial y^2} + \frac{Aq}{(1+Aqy)}\frac{\partial\bar{\theta}}{\partial y} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\bar{\theta}}{\partial y}] + \bar{K}[\frac{\partial^2\theta^*}{\partial y^2}$$

$$+ \frac{Aq}{(1+Aqy)}\frac{\partial\theta^*}{\partial y} + \frac{1}{(1+Aqy)^2}\frac{\partial^2\theta^*}{\partial x^{*2}} \pm \frac{J\sin\alpha}{(j\pm Jy\sin\alpha)}\frac{\partial\theta^*}{\partial y} + \frac{1}{(j\pm Jy\sin\alpha)^2}\frac{\partial^2\theta^*}{\partial z^2}]$$

$$+ \frac{\partial K^*}{\partial y}\frac{\partial\bar{\theta}}{\partial y} + \frac{\partial\bar{K}}{\partial y}\frac{\partial\theta^*}{\partial y}\}$$

The function $H(\omega)$ is the same function used in appendix A. For the present work, this function is

$$H(\omega) = \frac{2}{5}(1+\omega^2).$$

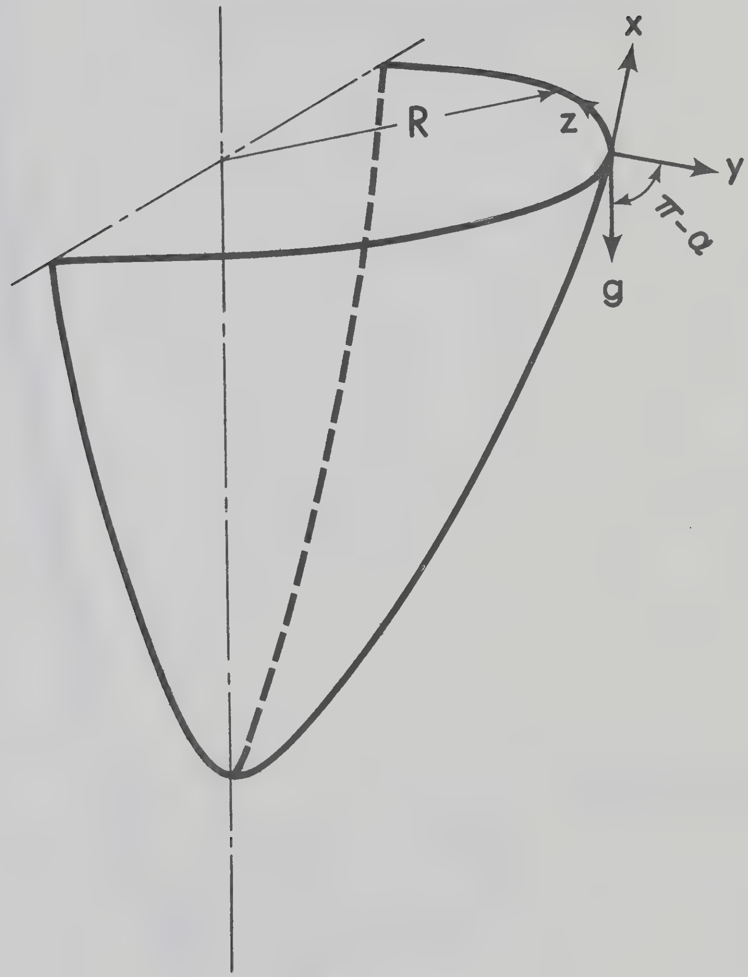


FIGURE 1. COORDINATE SYSTEM

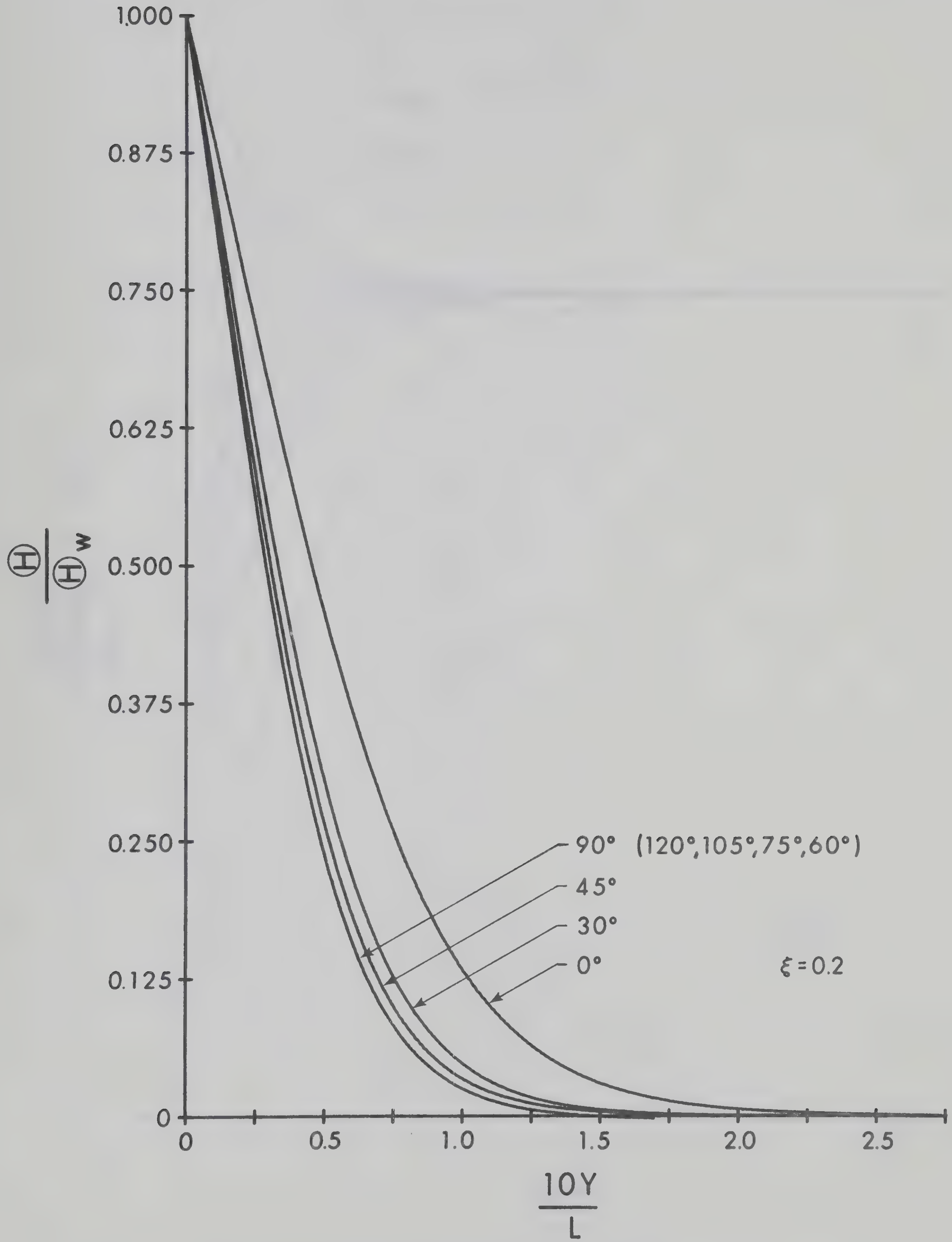


FIGURE 2a. EFFECT OF SURFACE INCLINATION ON TEMPERATURE PROFILES

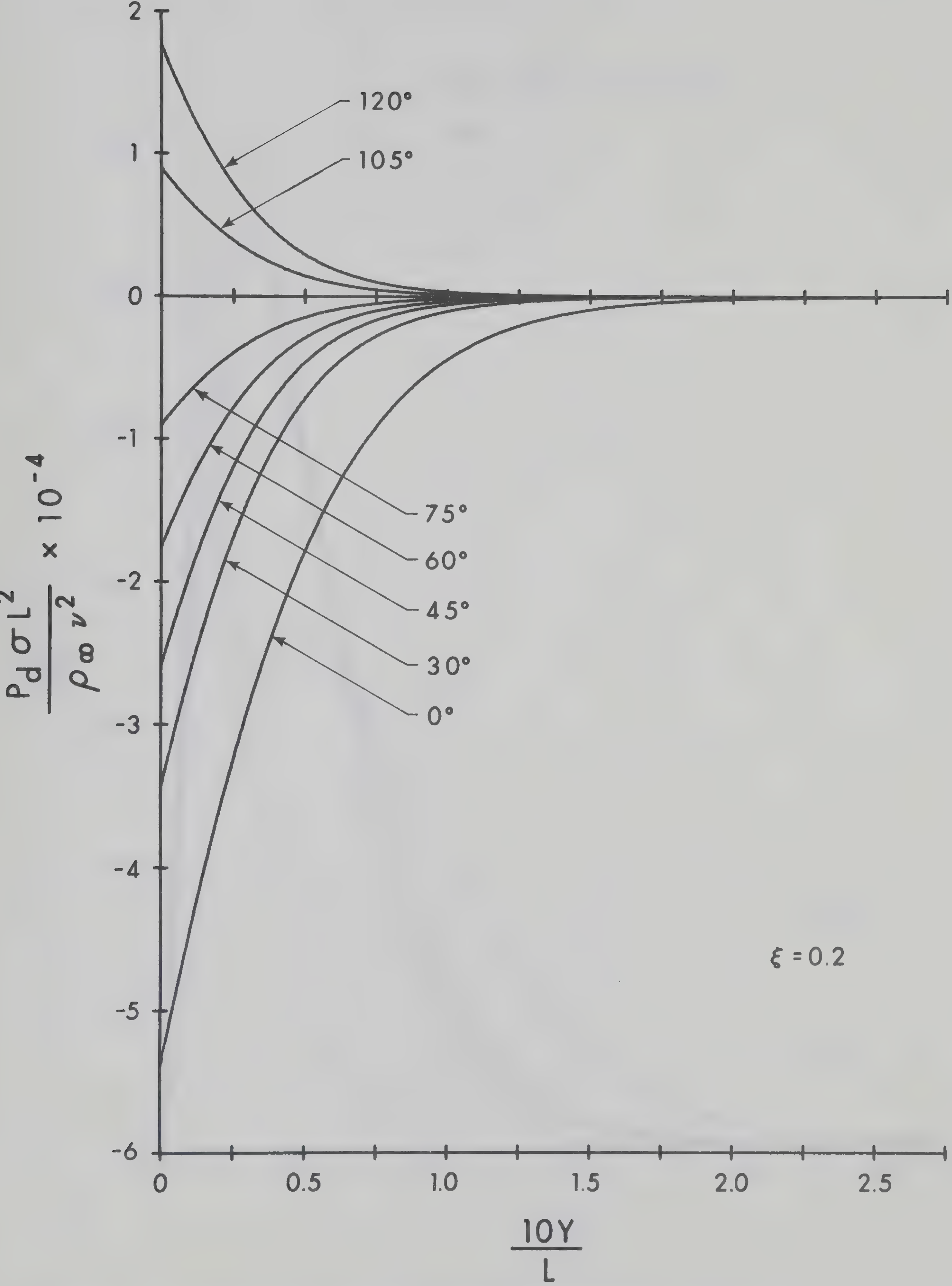


FIGURE 2b. EFFECT OF SURFACE INCLINATION ON PRESSURE PROFILES

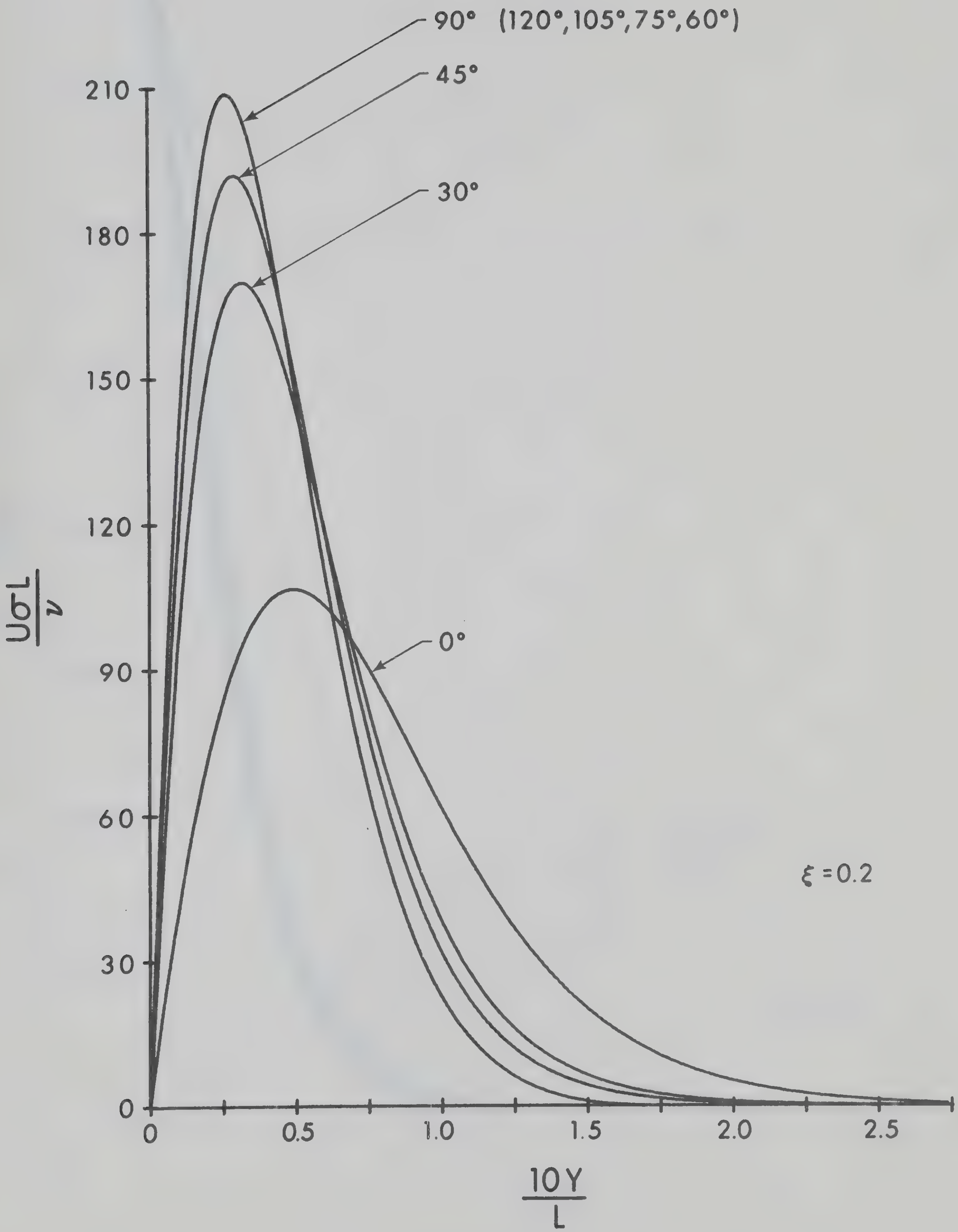


FIGURE 2c. EFFECT OF SURFACE INCLINATION ON VELOCITY PROFILES

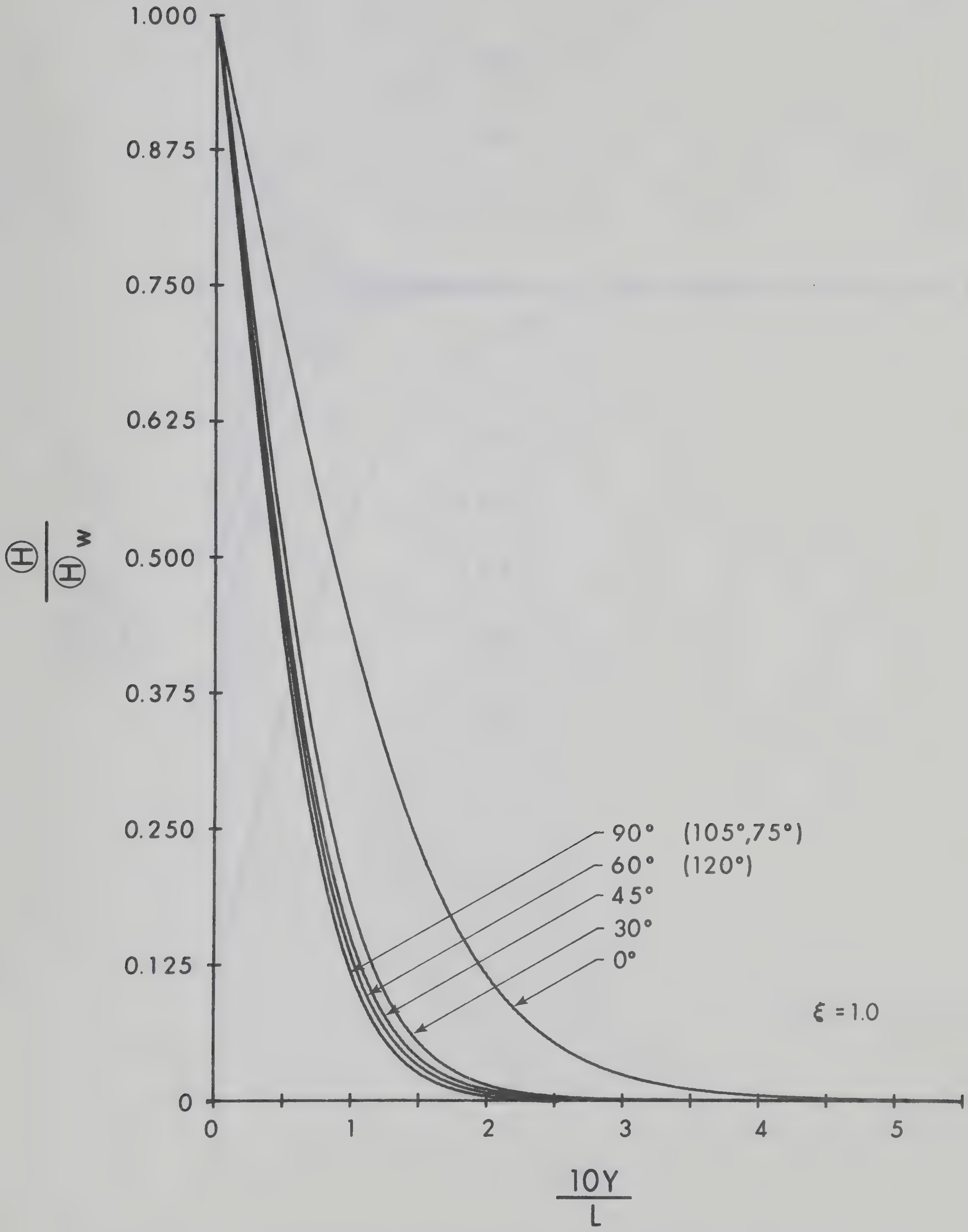


FIGURE 3a. EFFECT OF SURFACE INCLINATION ON TEMPERATURE PROFILES

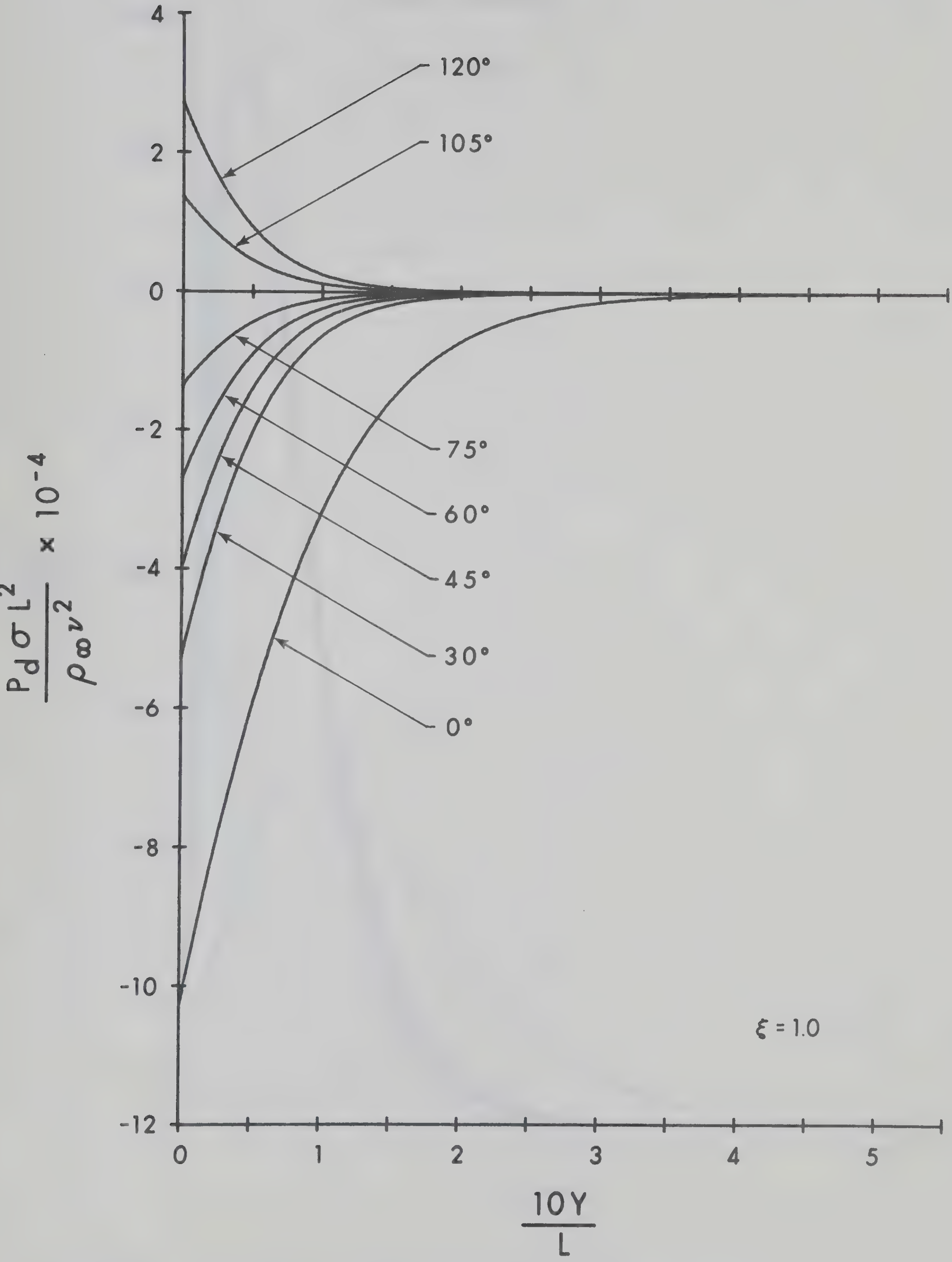


FIGURE 3b. EFFECT OF SURFACE INCLINATION ON PRESSURE PROFILES

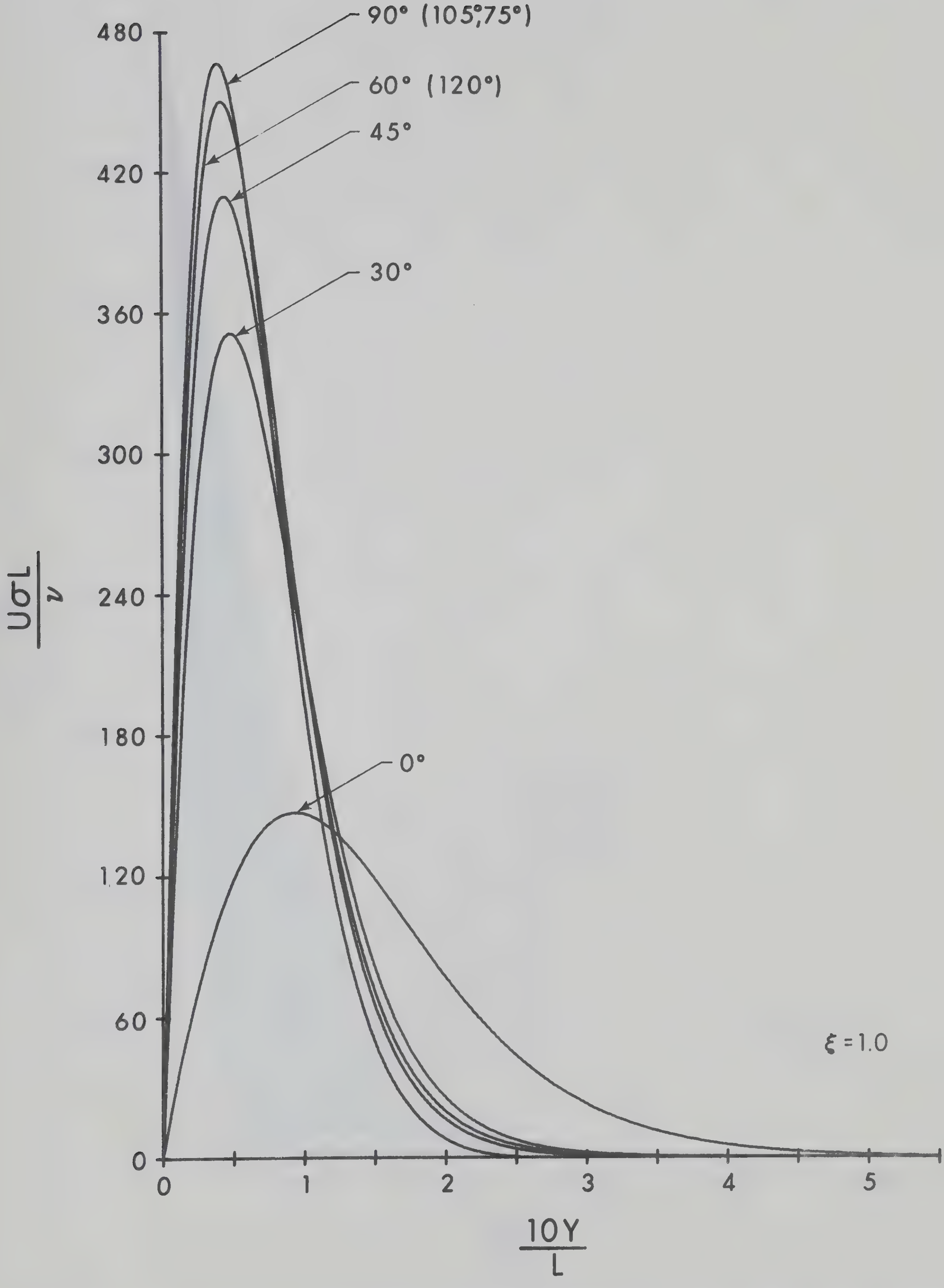


FIGURE 3c. EFFECT OF SURFACE INCLINATION ON VELOCITY PROFILES

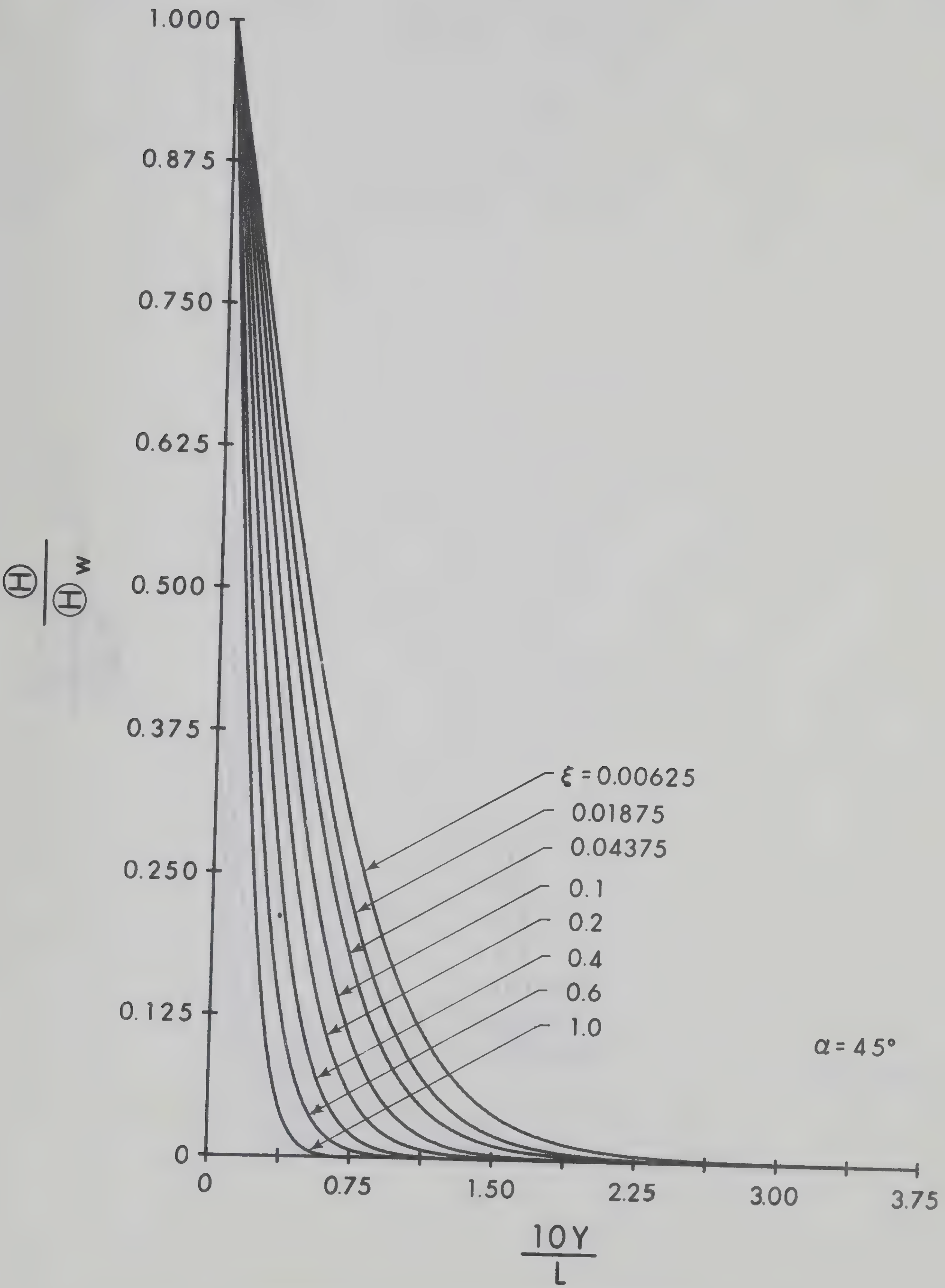


FIGURE 4a. VARIATION WITH POSITION ALONG THE SURFACE OF TEMPERATURE PROFILES

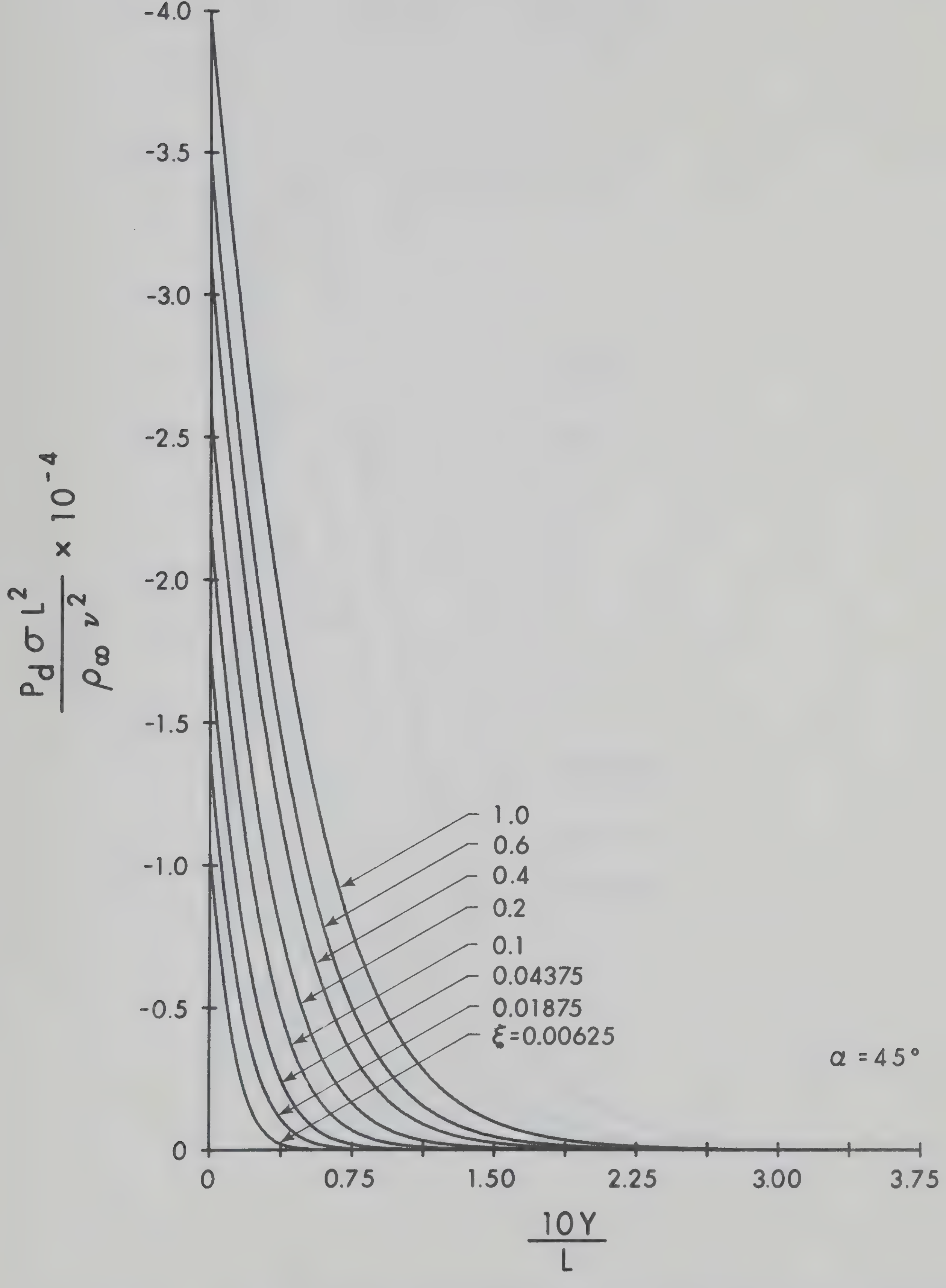


FIGURE 4b. VARIATION WITH POSITION ALONG THE SURFACE OF PRESSURE PROFILES

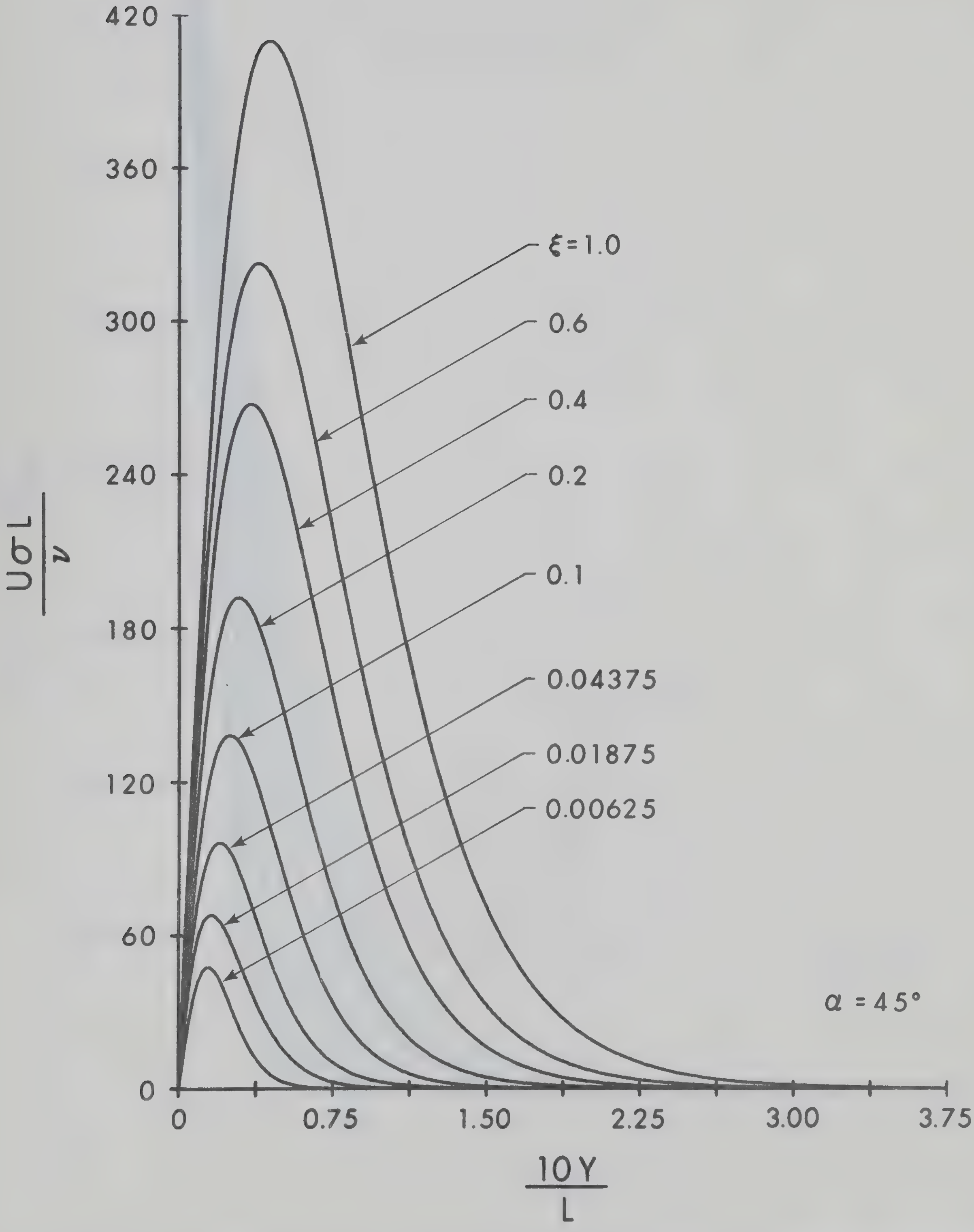
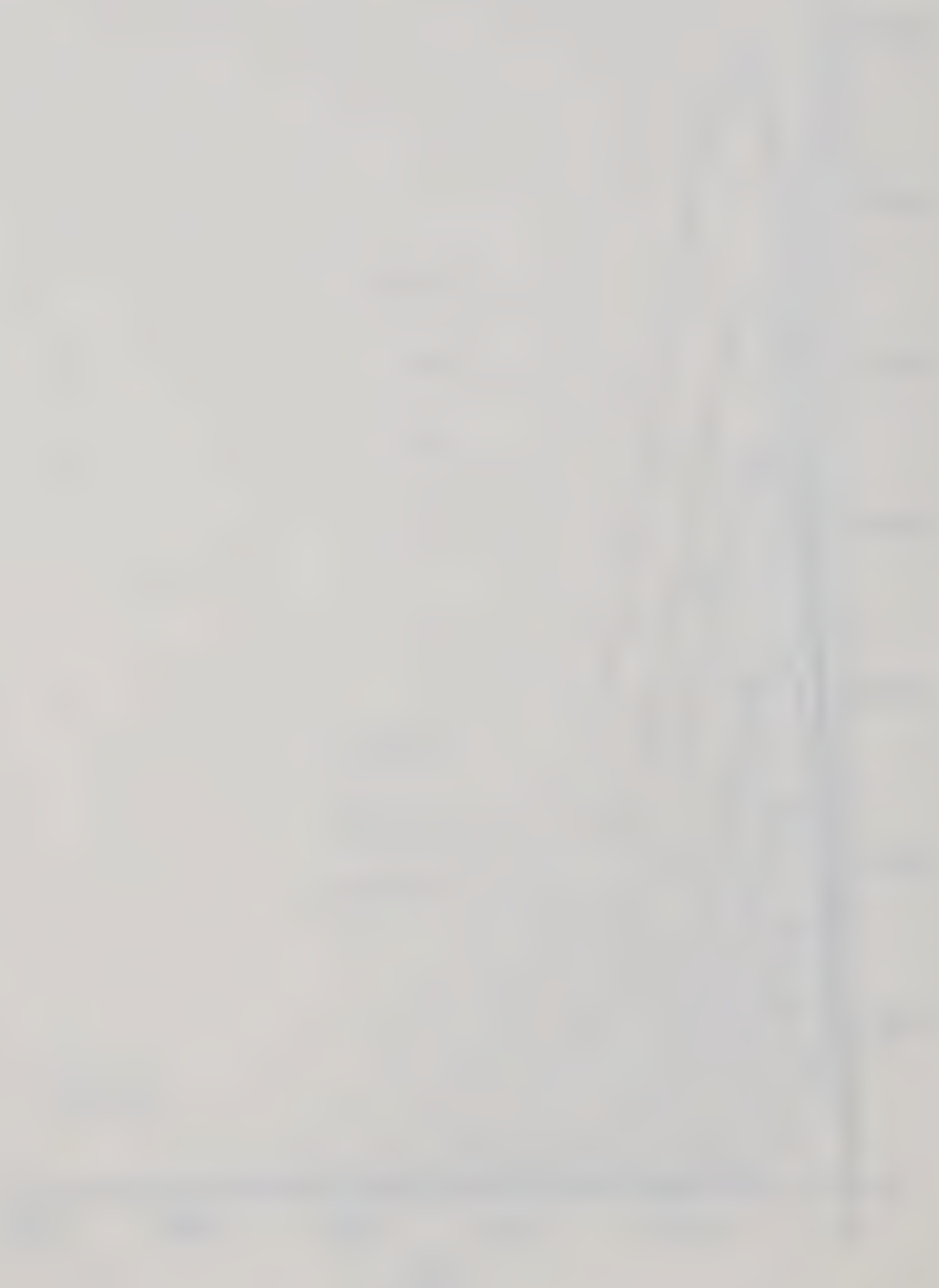


FIGURE 4c. VARIATION WITH POSITION ALONG THE SURFACE OF VELOCITY PROFILES



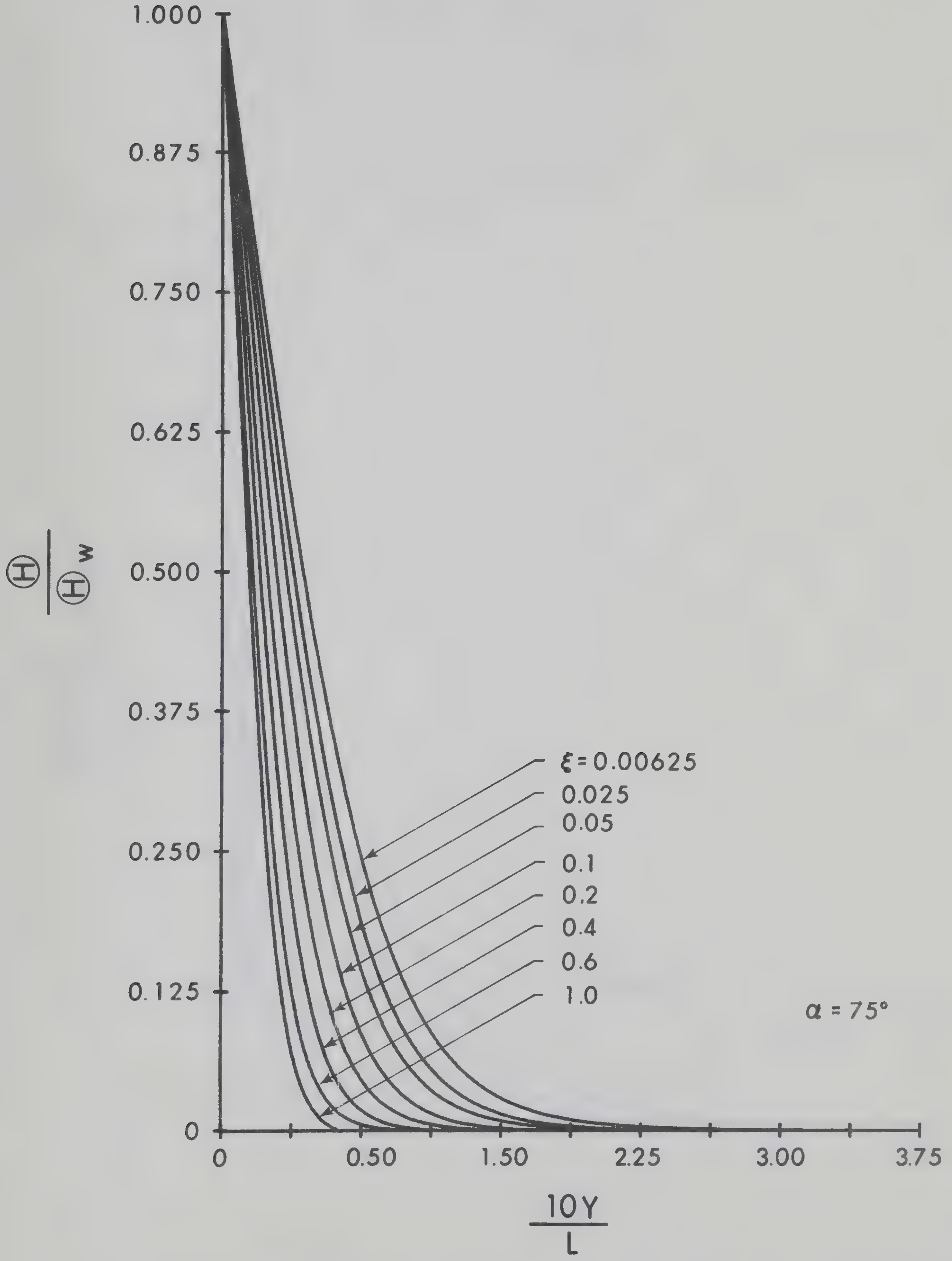


FIGURE 5a. VARIATION WITH POSITION ALONG THE SURFACE OF TEMPERATURE PROFILES

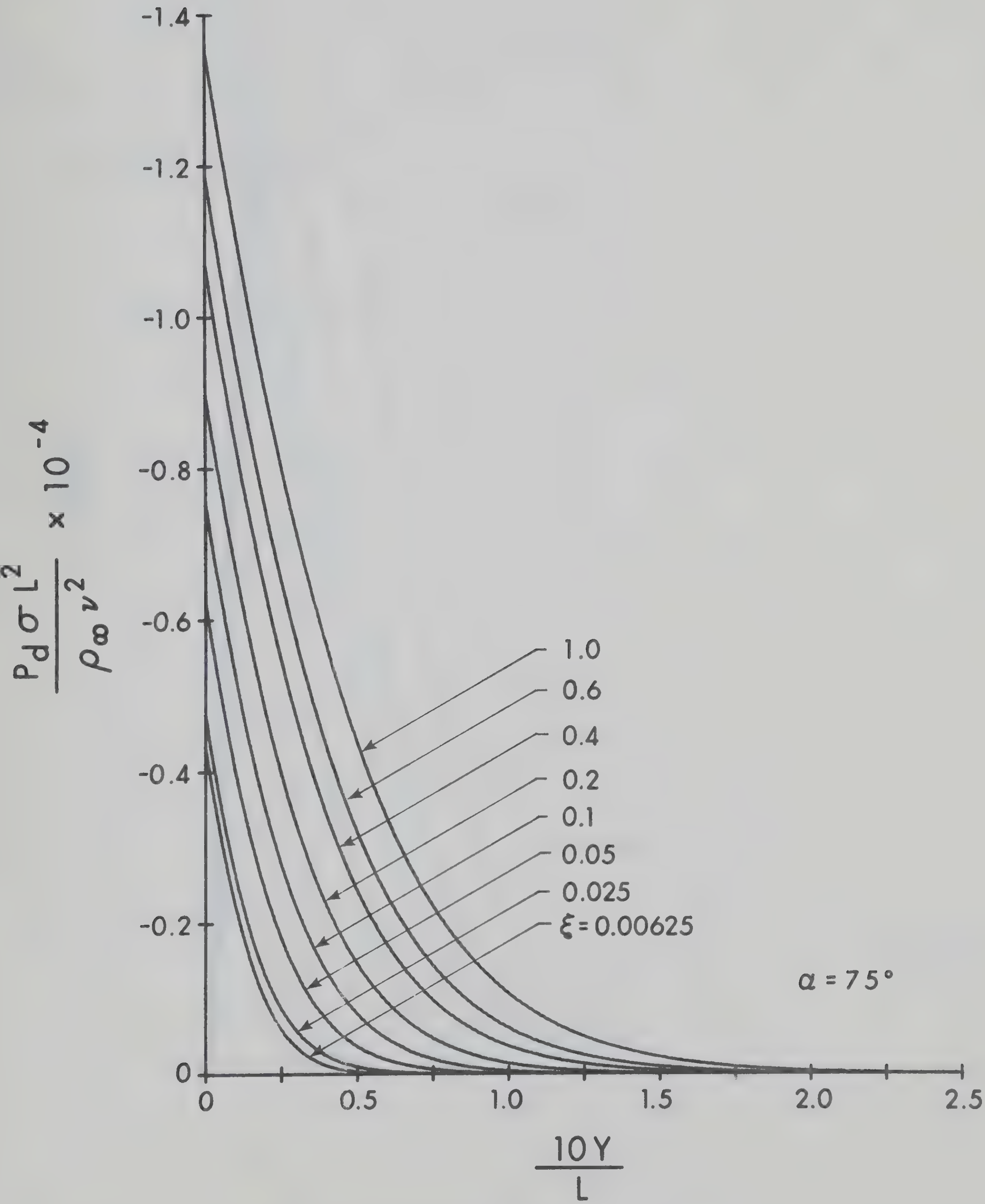


FIGURE 5b. VARIATION WITH POSITION ALONG THE SURFACE OF PRESSURE PROFILES

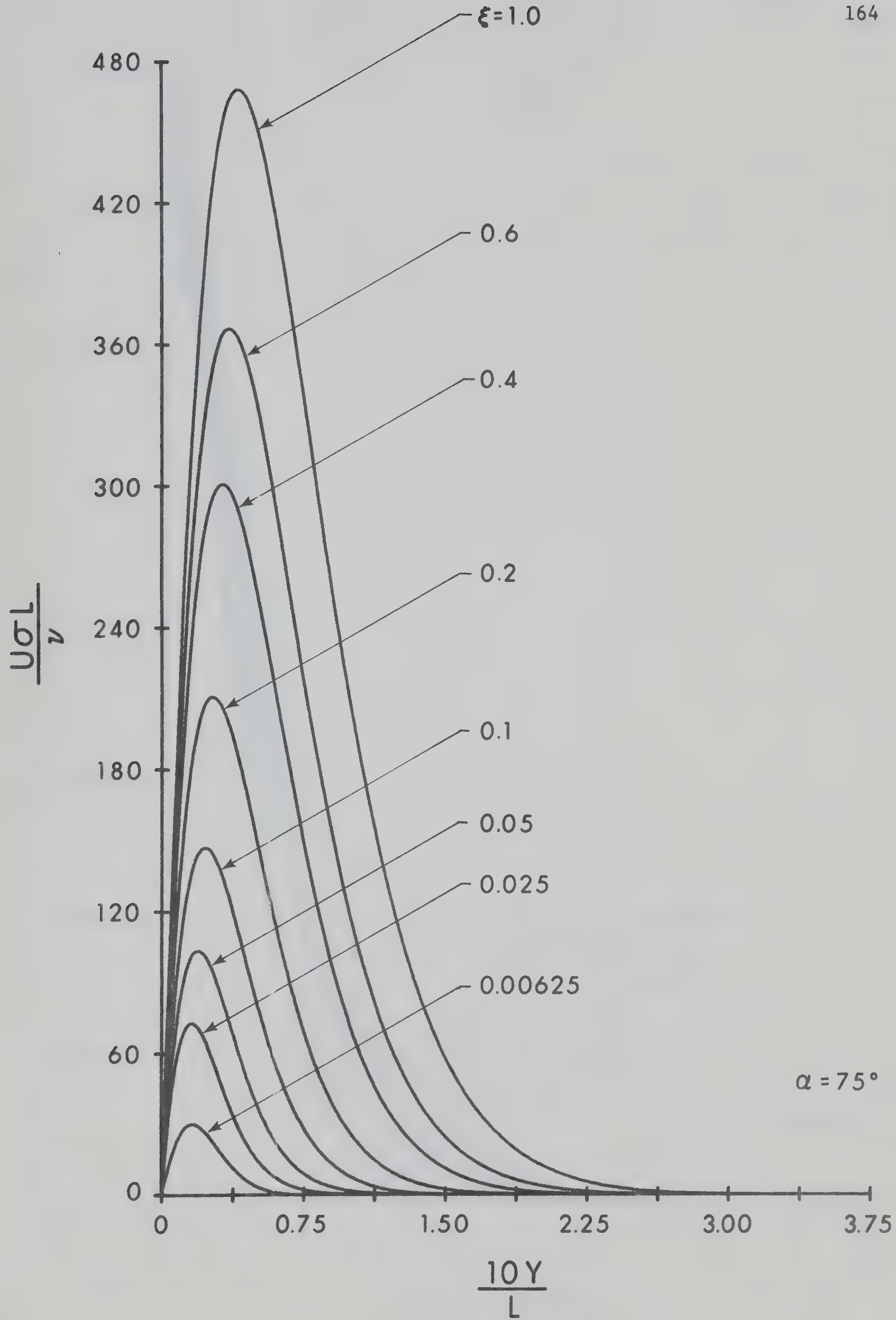
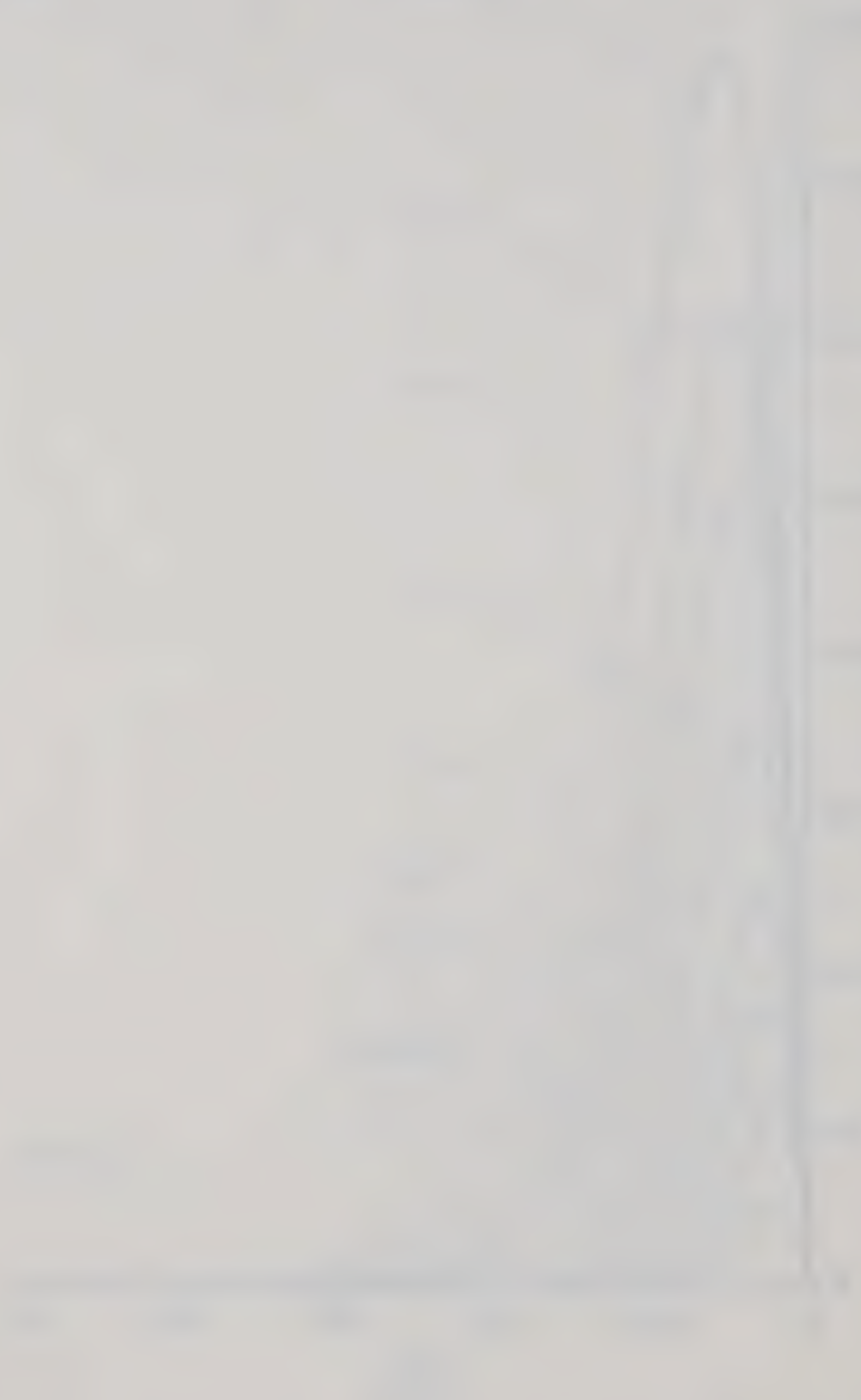


FIGURE 5c. VARIATION WITH POSITION ALONG THE SURFACE OF VELOCITY PROFILES



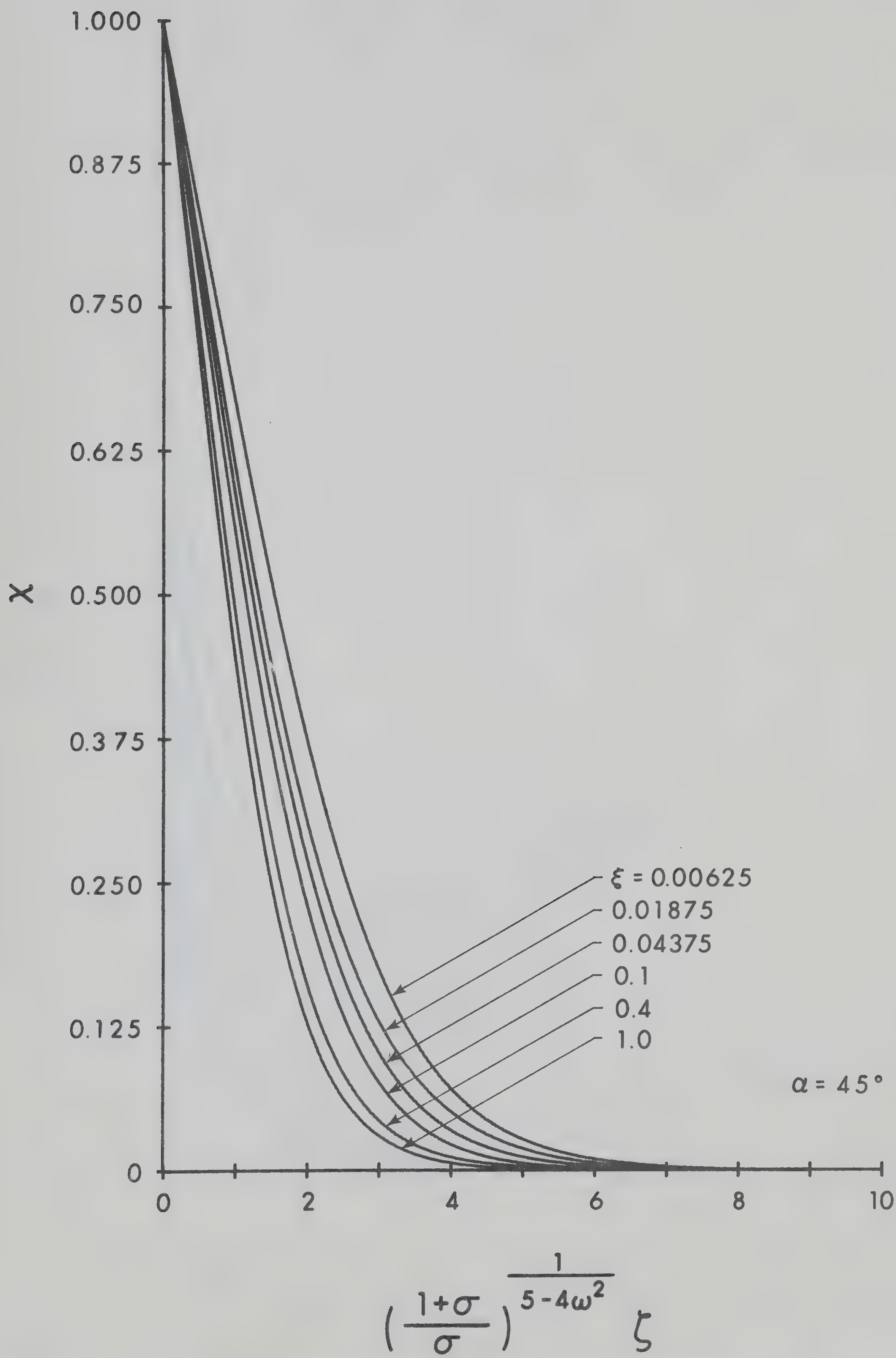
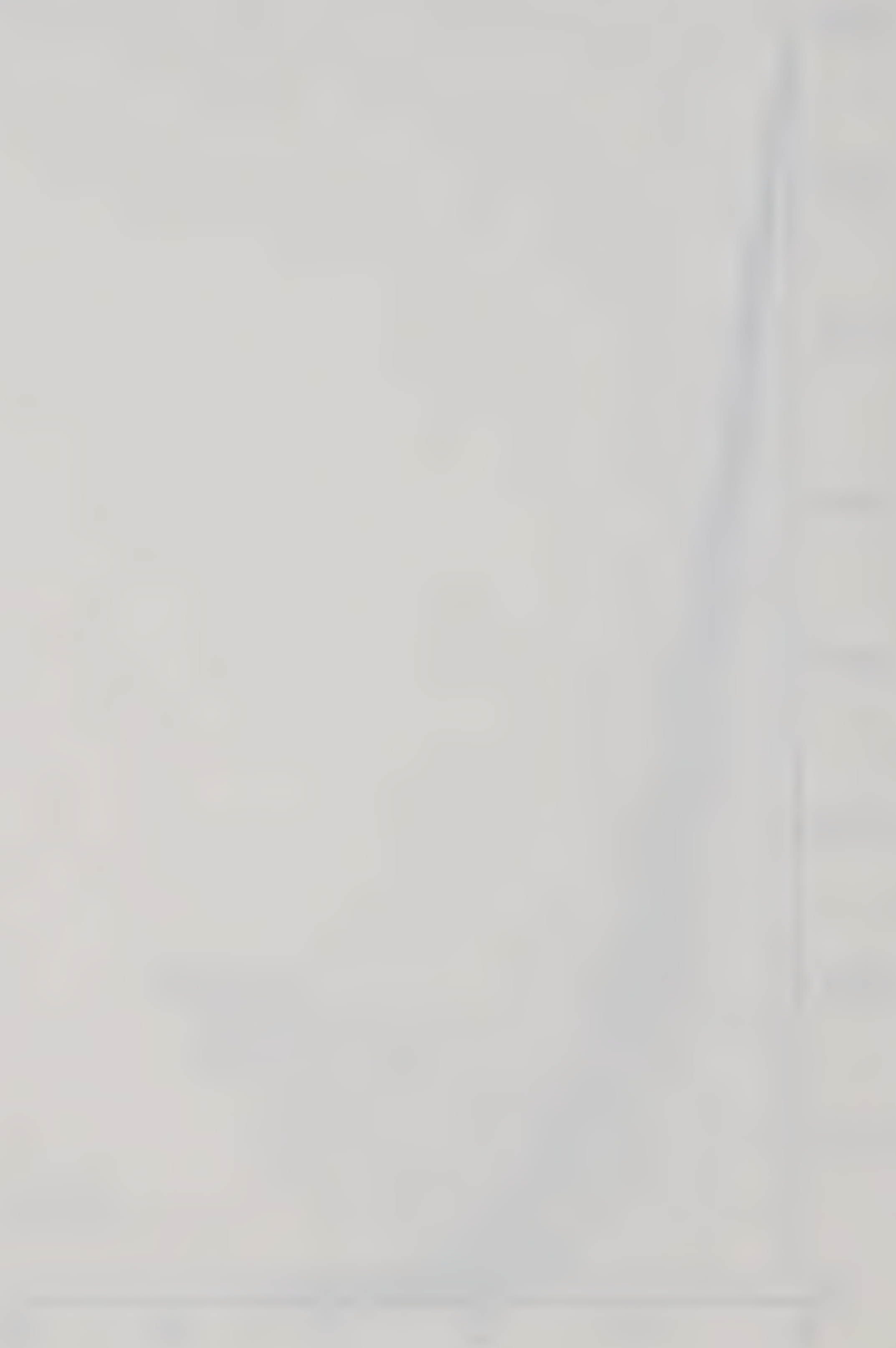


FIGURE 6a. DEPARTURE FROM SIMILARITY OF TEMPERATURE PROFILES



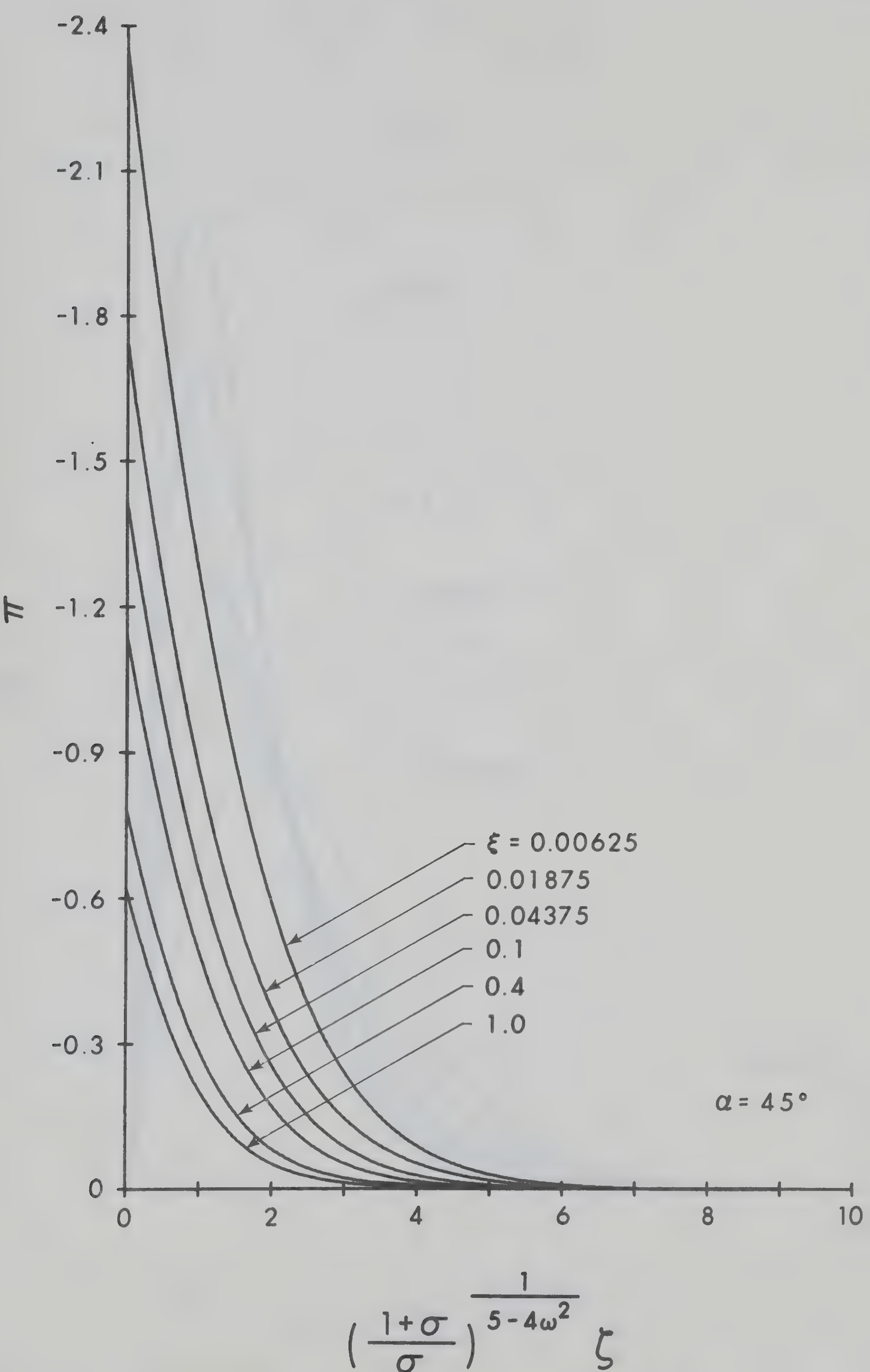
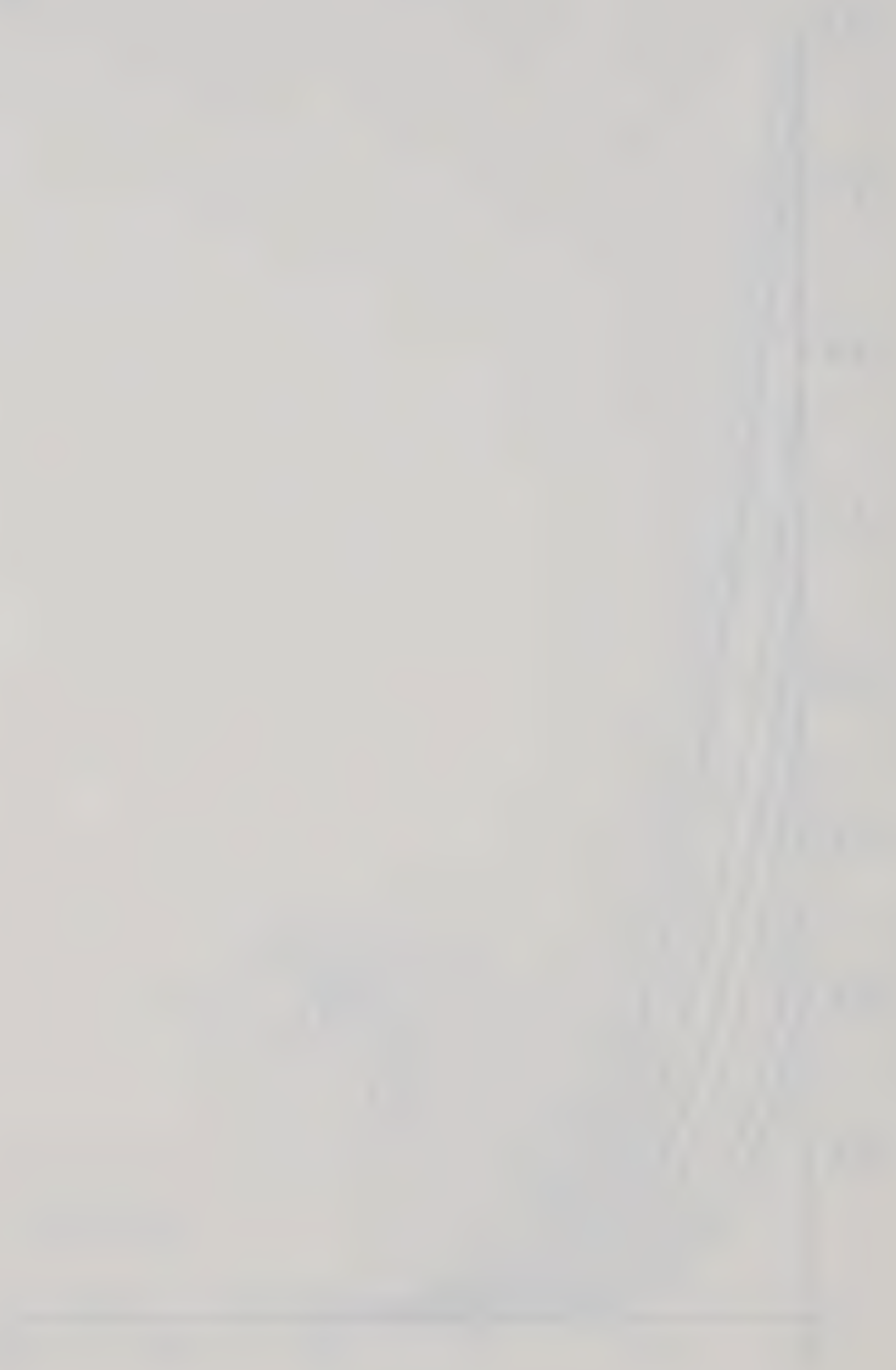


FIGURE 6b. DEPARTURE FROM SIMILARITY OF PRESSURE PROFILES



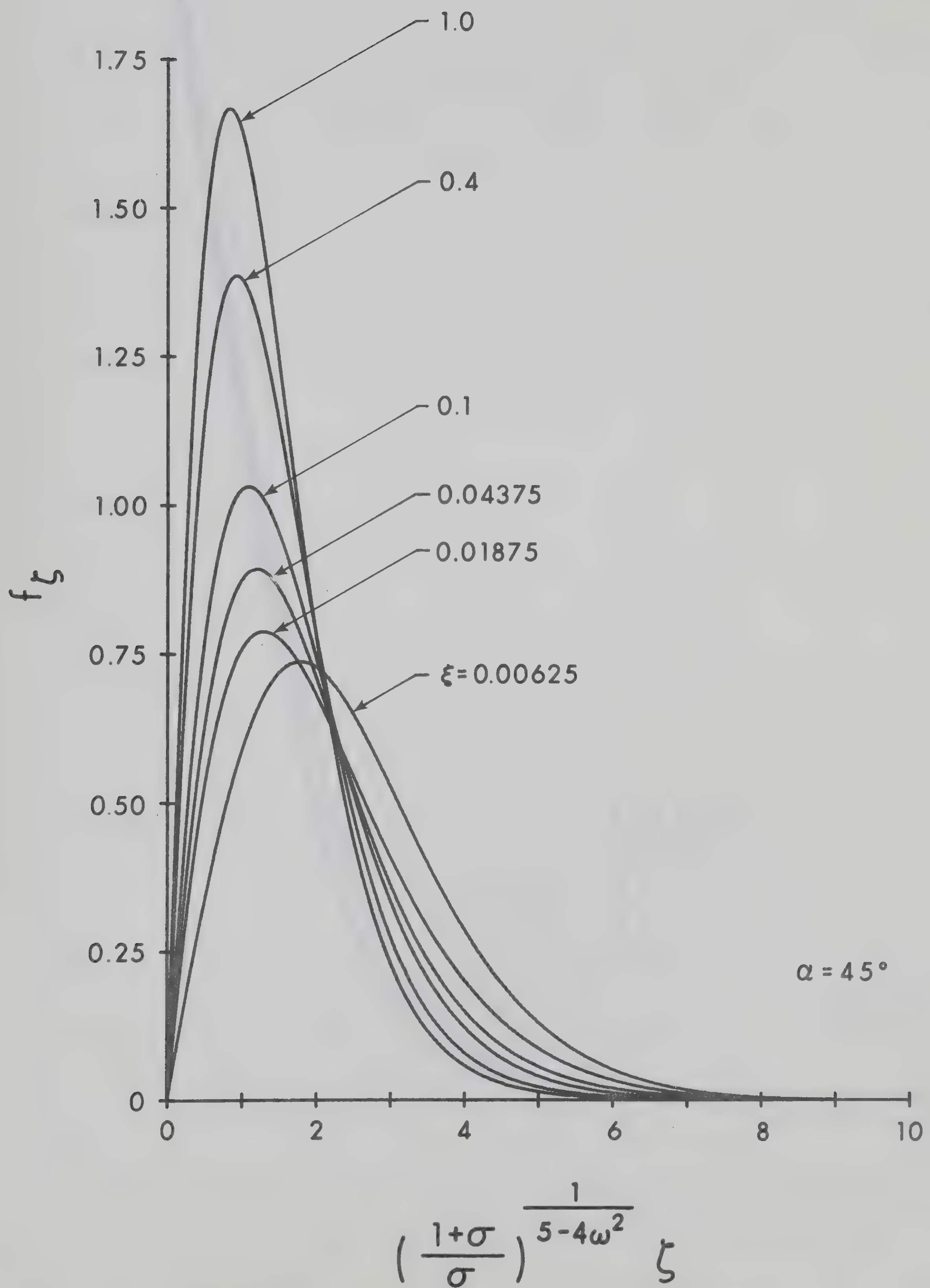
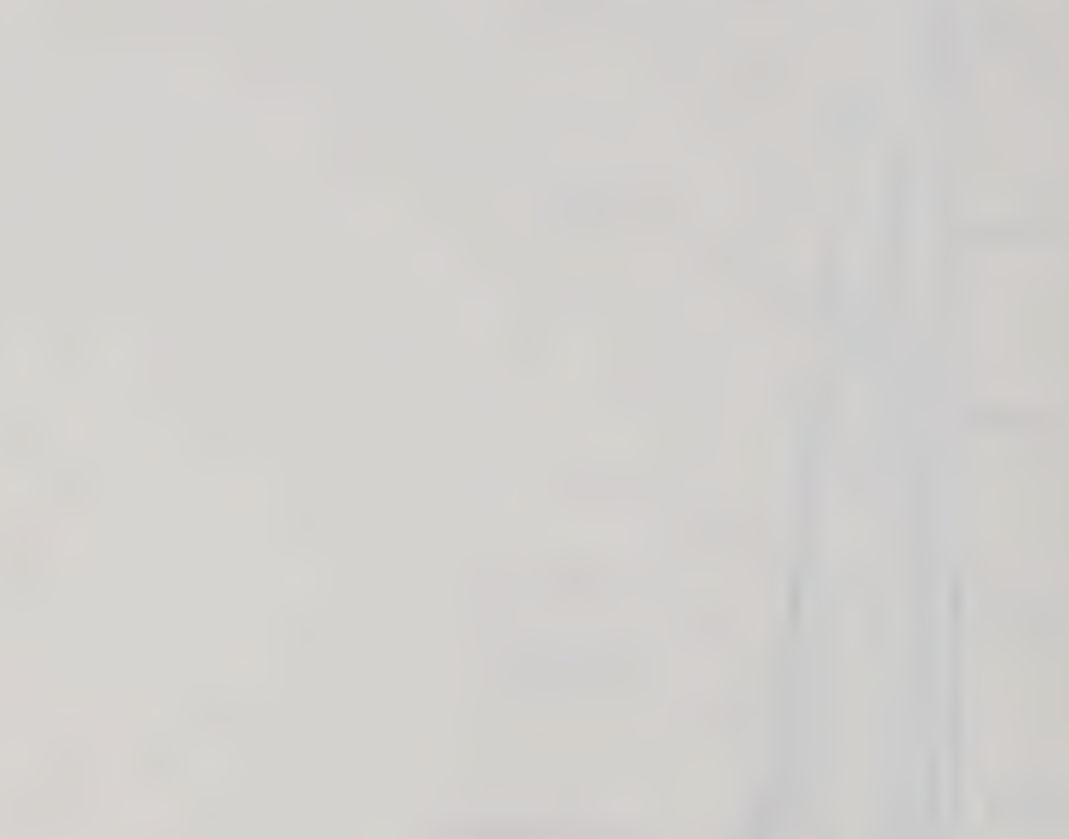


FIGURE 6c. DEPARTURE FROM SIMILARITY OF VELOCITY PROFILES



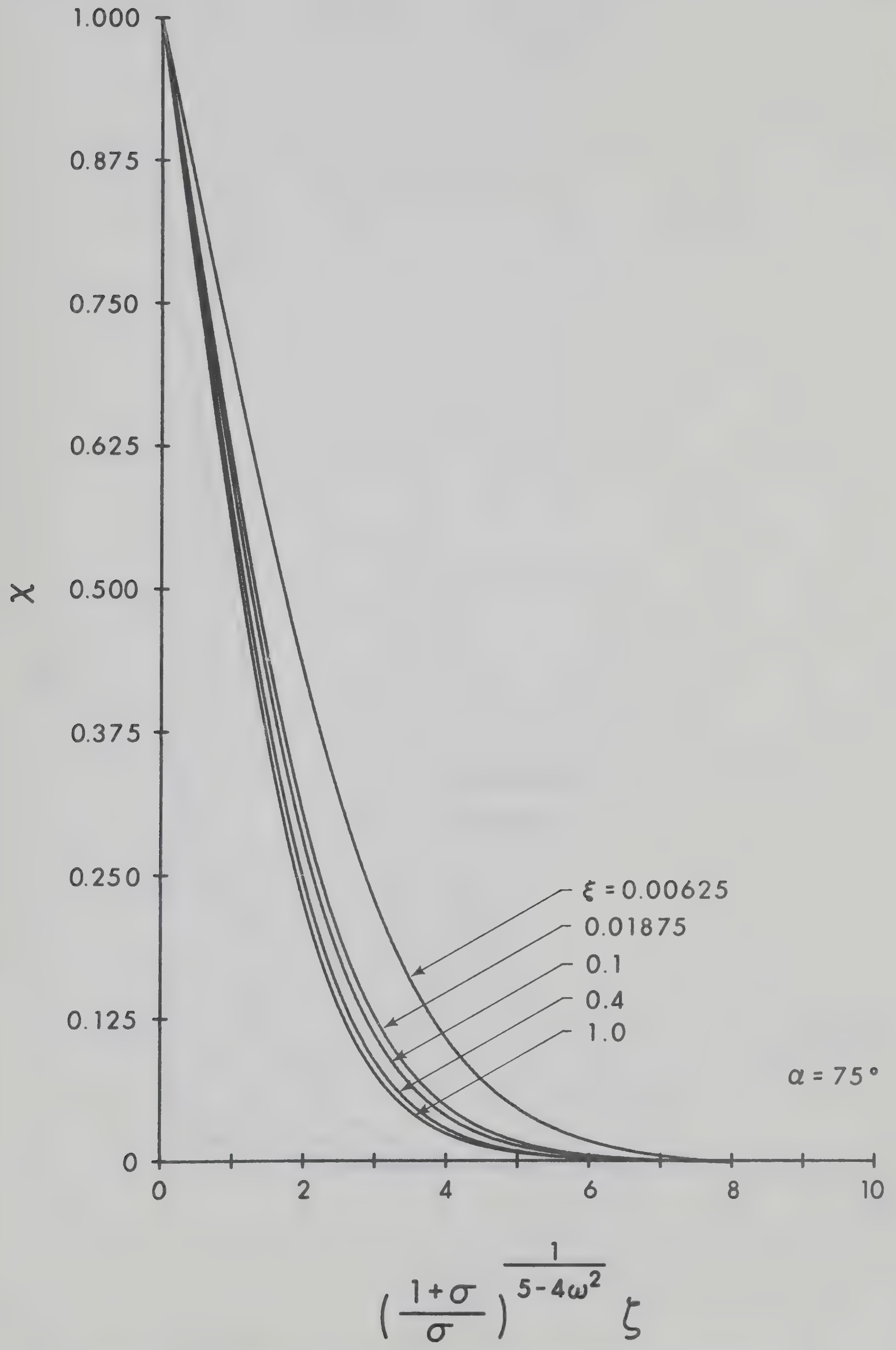


FIGURE 7a. DEPARTURE FROM SIMILARITY OF TEMPERATURE PROFILES

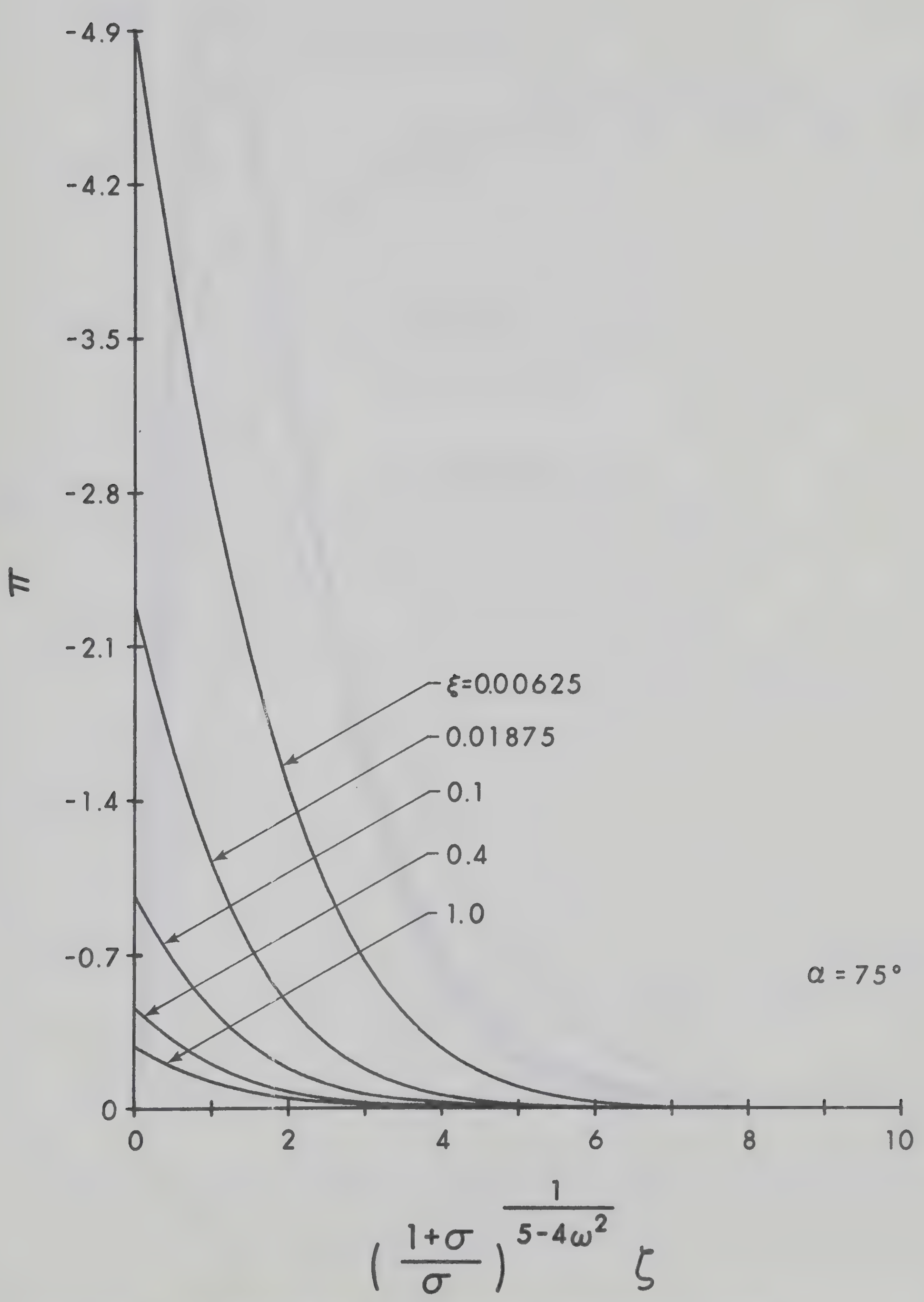


FIGURE 7b. DEPARTURE FROM SIMILARITY OF PRESSURE PROFILES

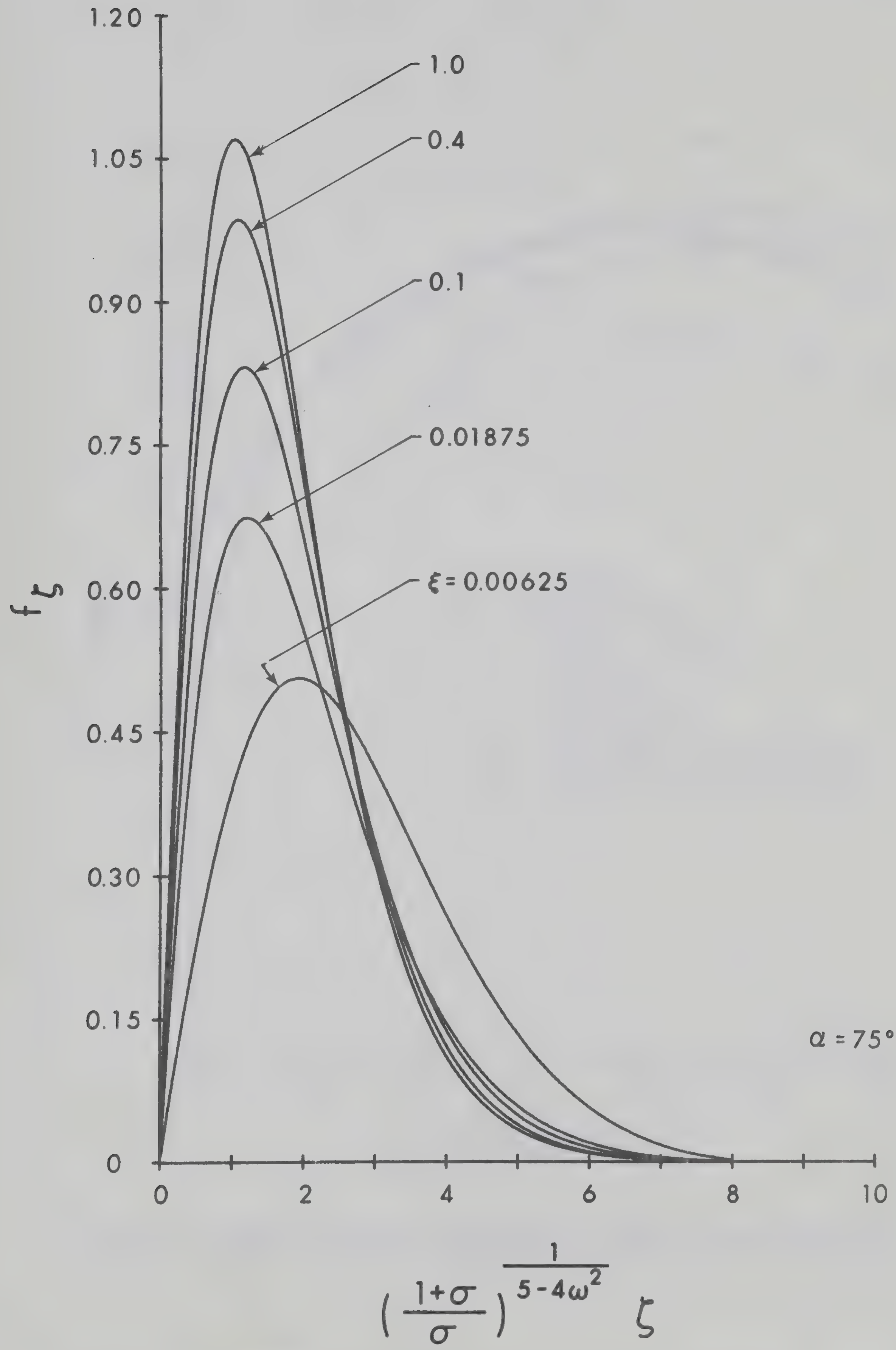


FIGURE 7c. DEPARTURE FROM SIMILARITY OF VELOCITY PROFILES

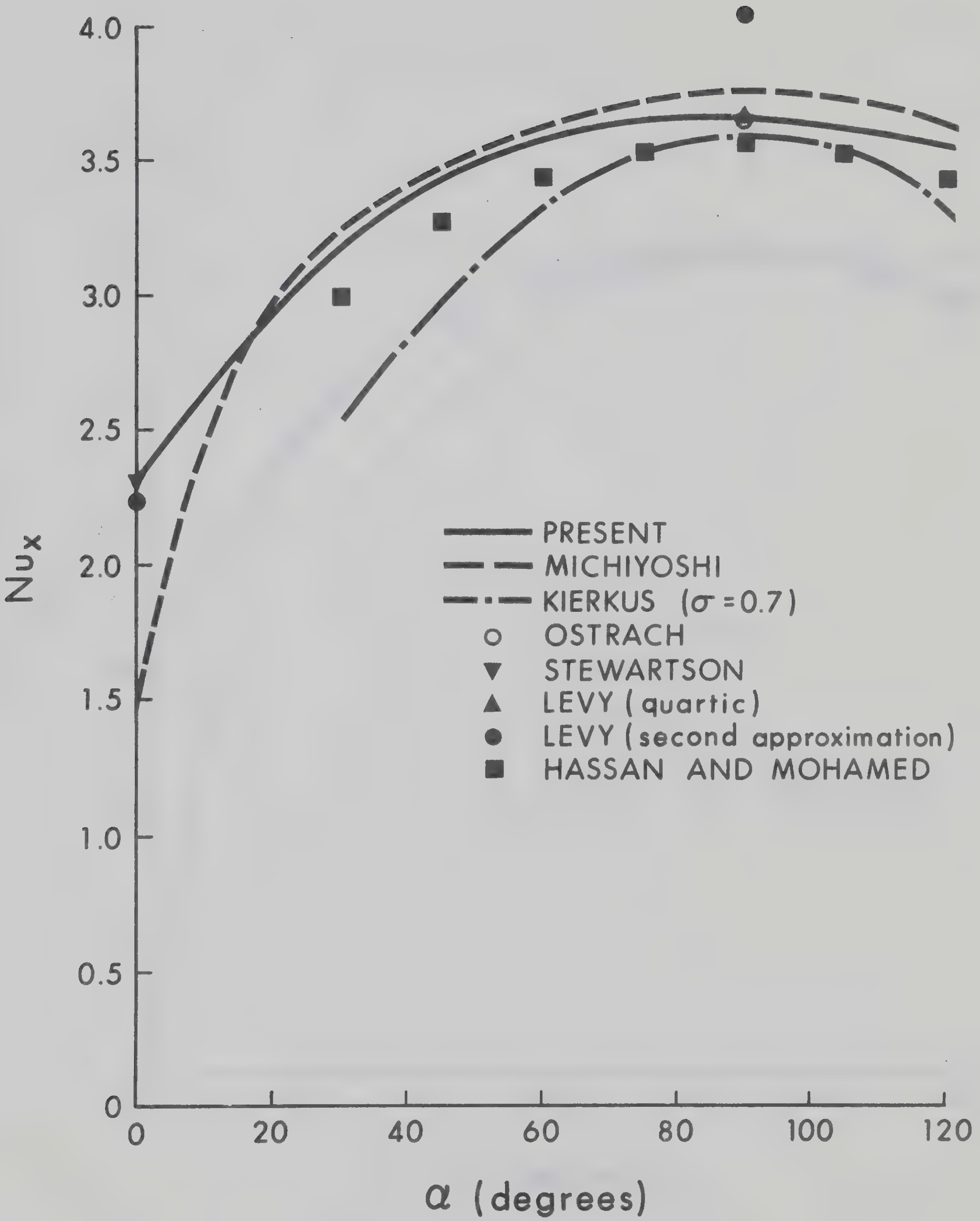


FIGURE 8. EFFECT OF SURFACE INCLINATION ON MEAN-FLOW HEAT TRANSFER ($\xi = 0.2$)

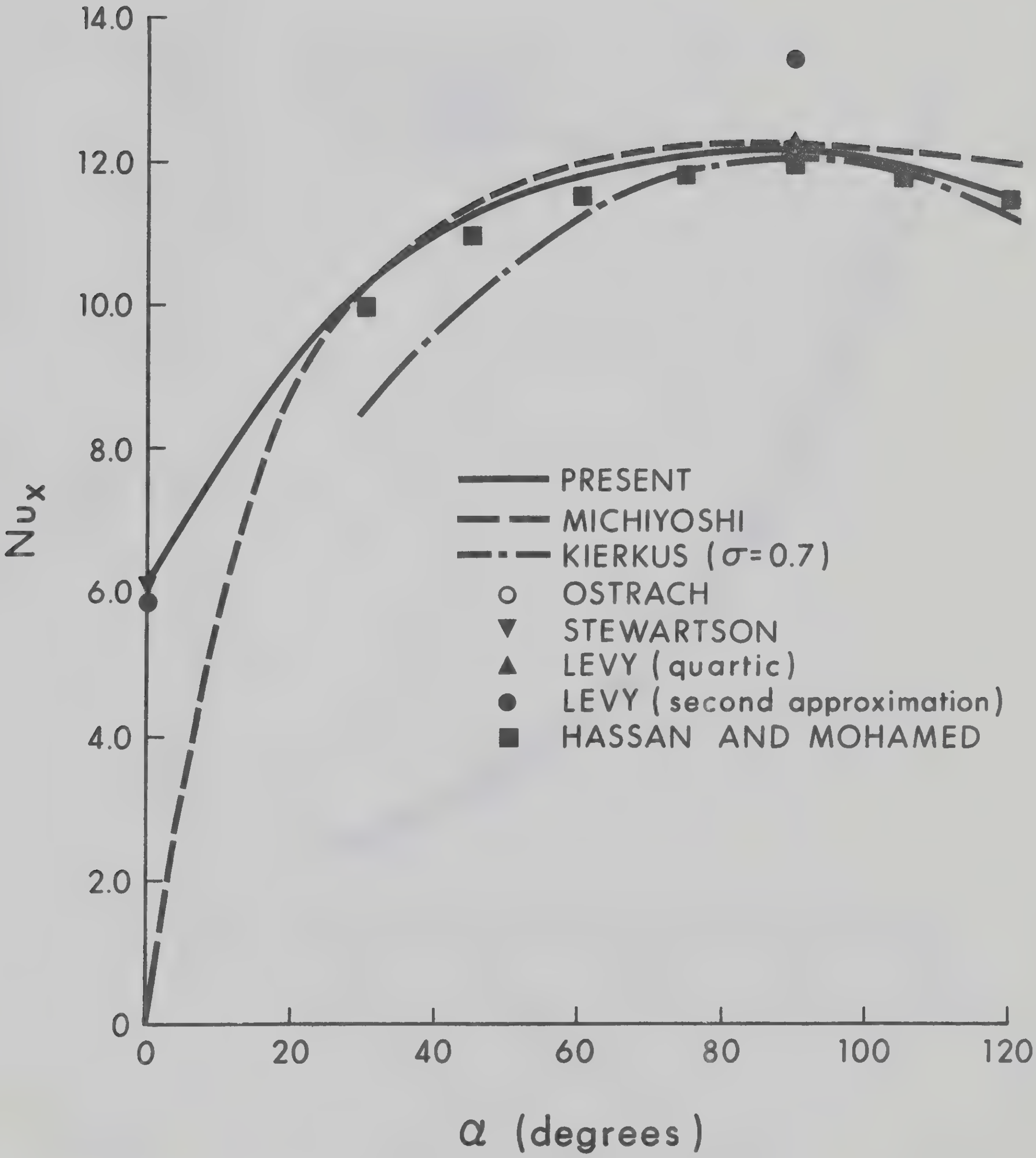


FIGURE 9. EFFECT OF SURFACE INCLINATION ON MEAN-FLOW HEAT TRANSFER ($\xi = 1.0$)

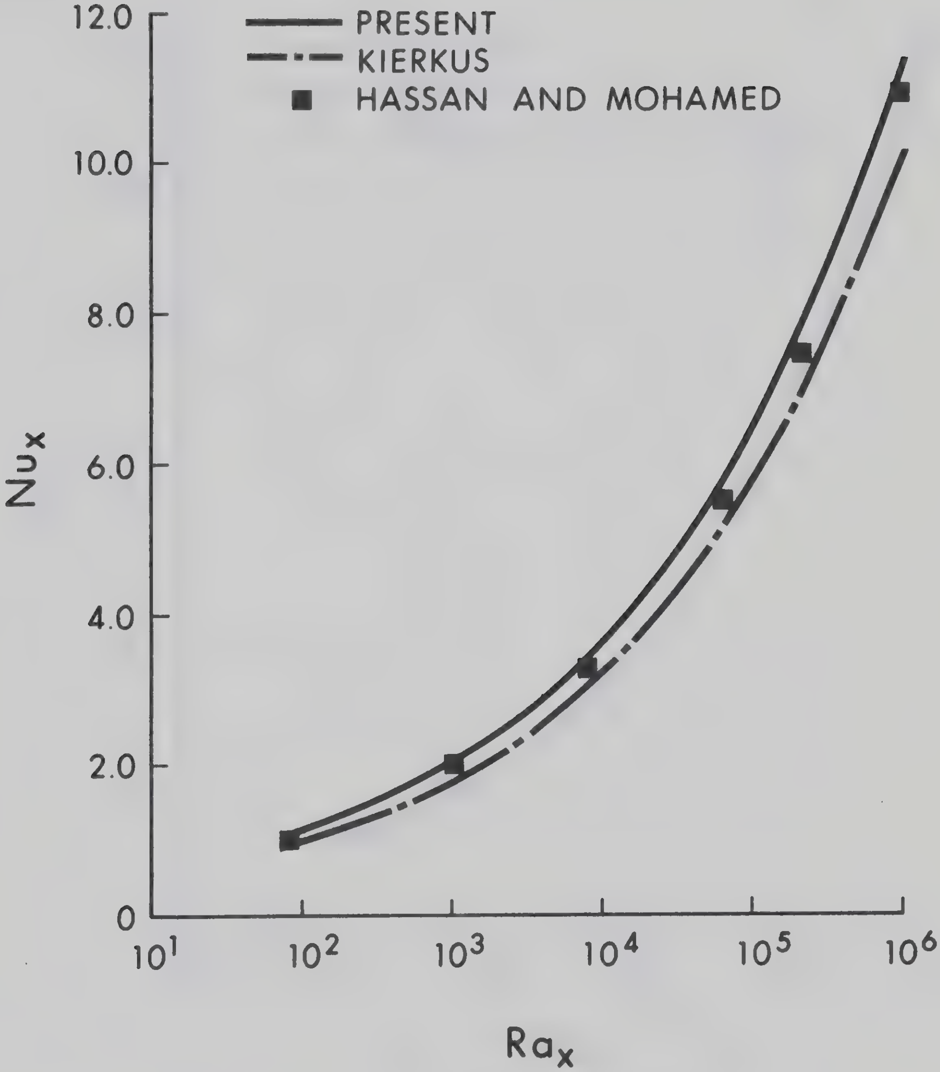


FIGURE 10. VARIATION WITH POSITION ALONG THE SURFACE OF MEAN-FLOW HEAT TRANSFER ($\alpha = 45°$)

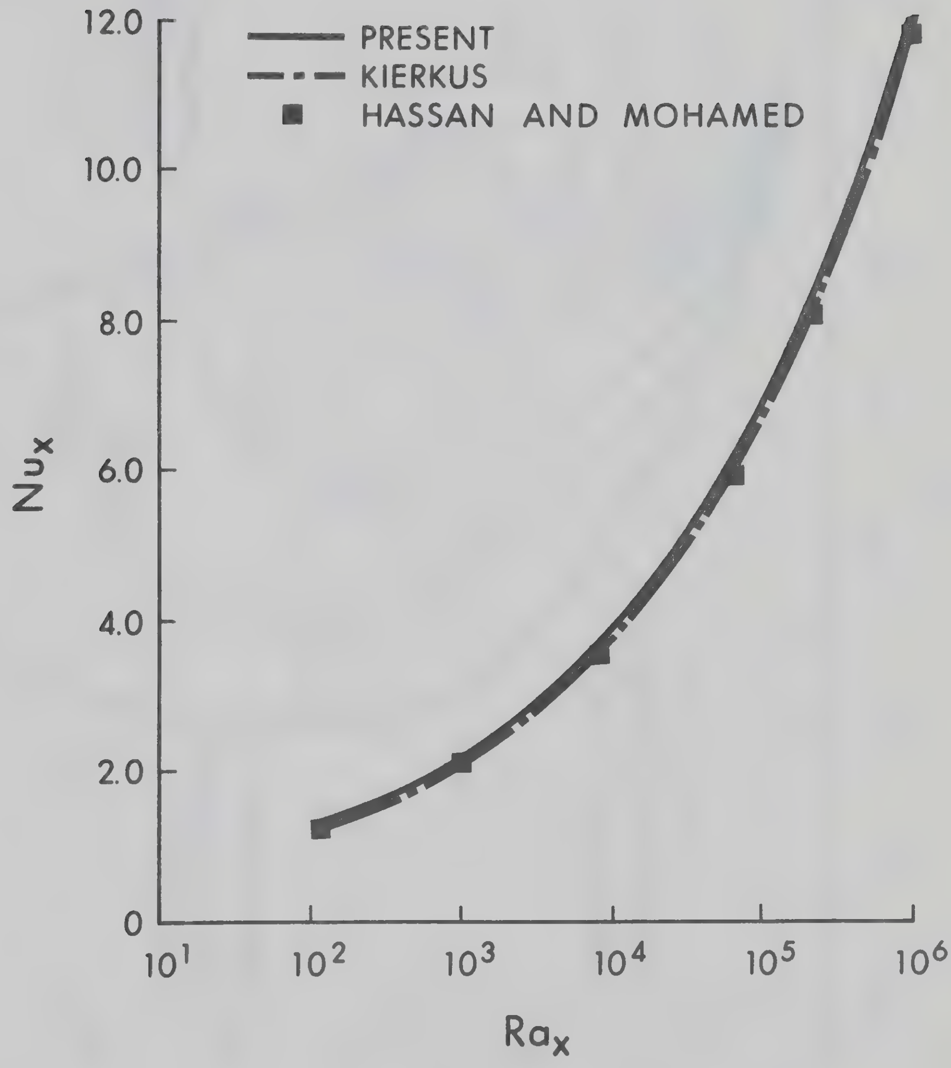
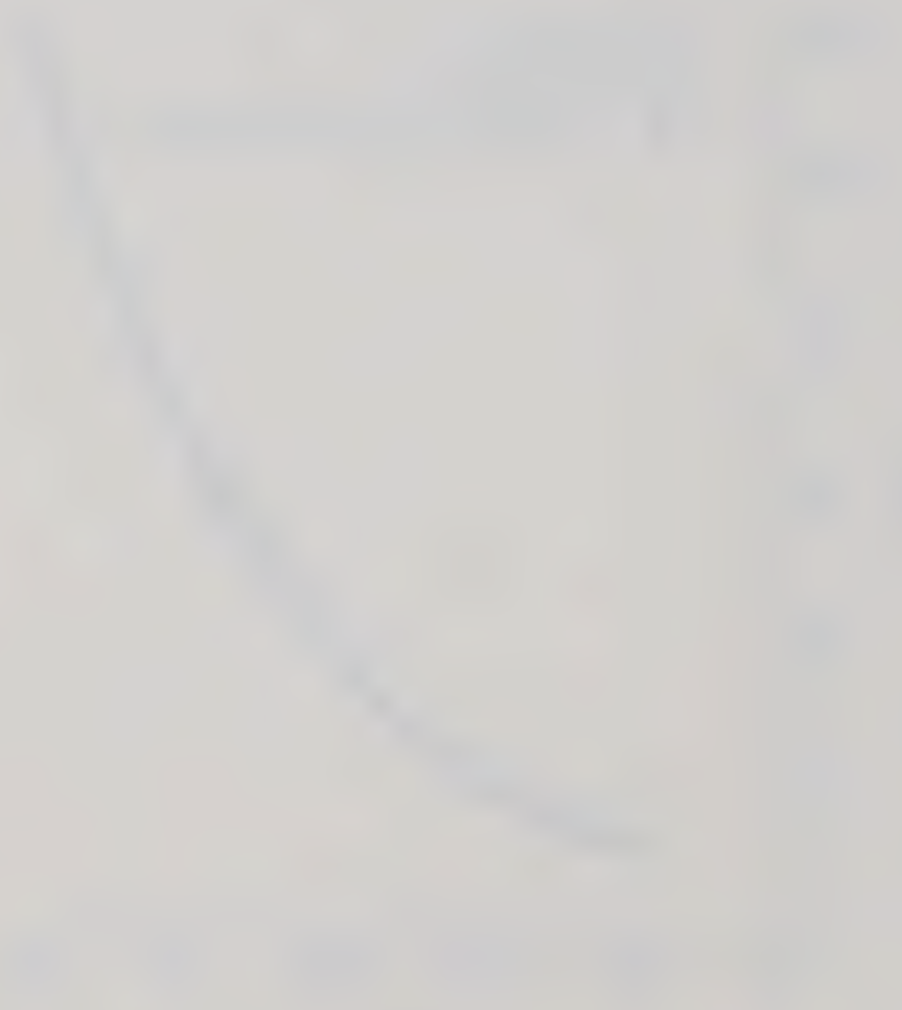


FIGURE 11. VARIATION WITH POSITION ALONG THE SURFACE OF MEAN-FLOW HEAT TRANSFER (α = 75°)



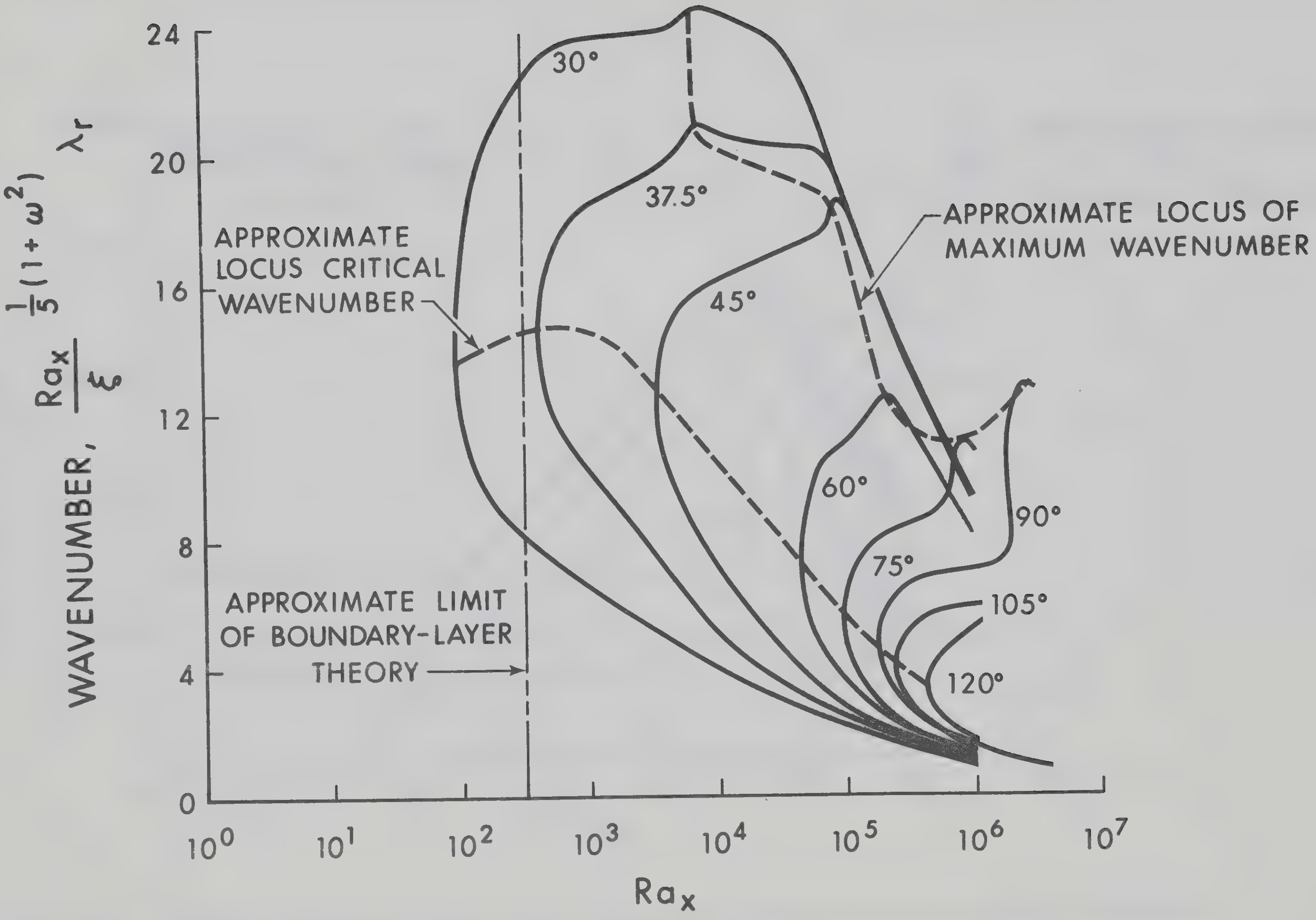


FIGURE 12a. NEUTRAL STABILITY CURVES FOR TOLLMIEN-SCHLICHTING WAVE DISTURBANCES (WAVENUMBER)

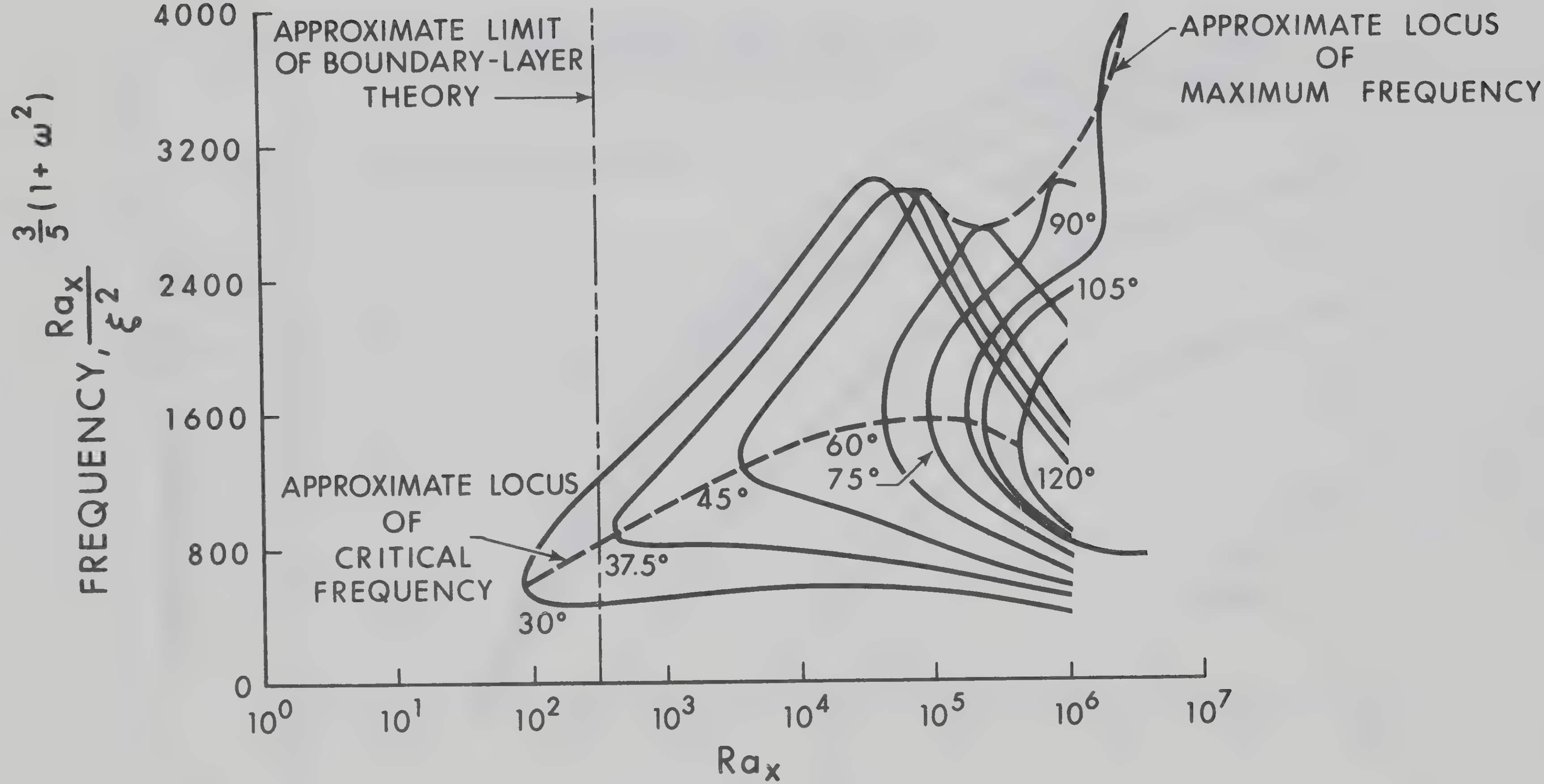


FIGURE 12b. NEUTRAL STABILITY CURVES FOR TOLLMIEN-SCHLICHTING WAVE DISTURBANCES (FREQUENCY)

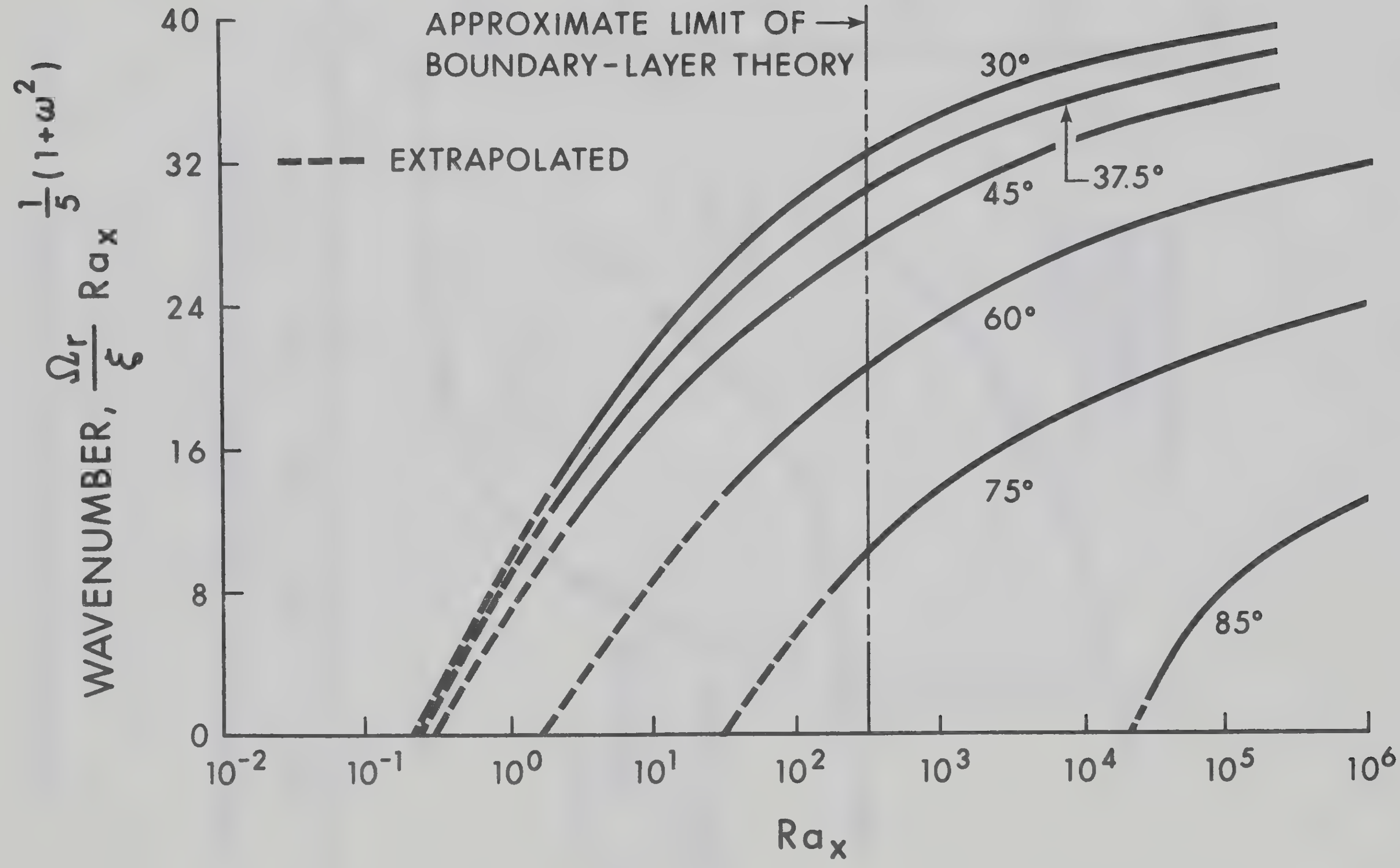


FIGURE 13. NEUTRAL STABILITY CURVES FOR TAYLOR-GOERTLER ROLL-VORTEX DISTURBANCES

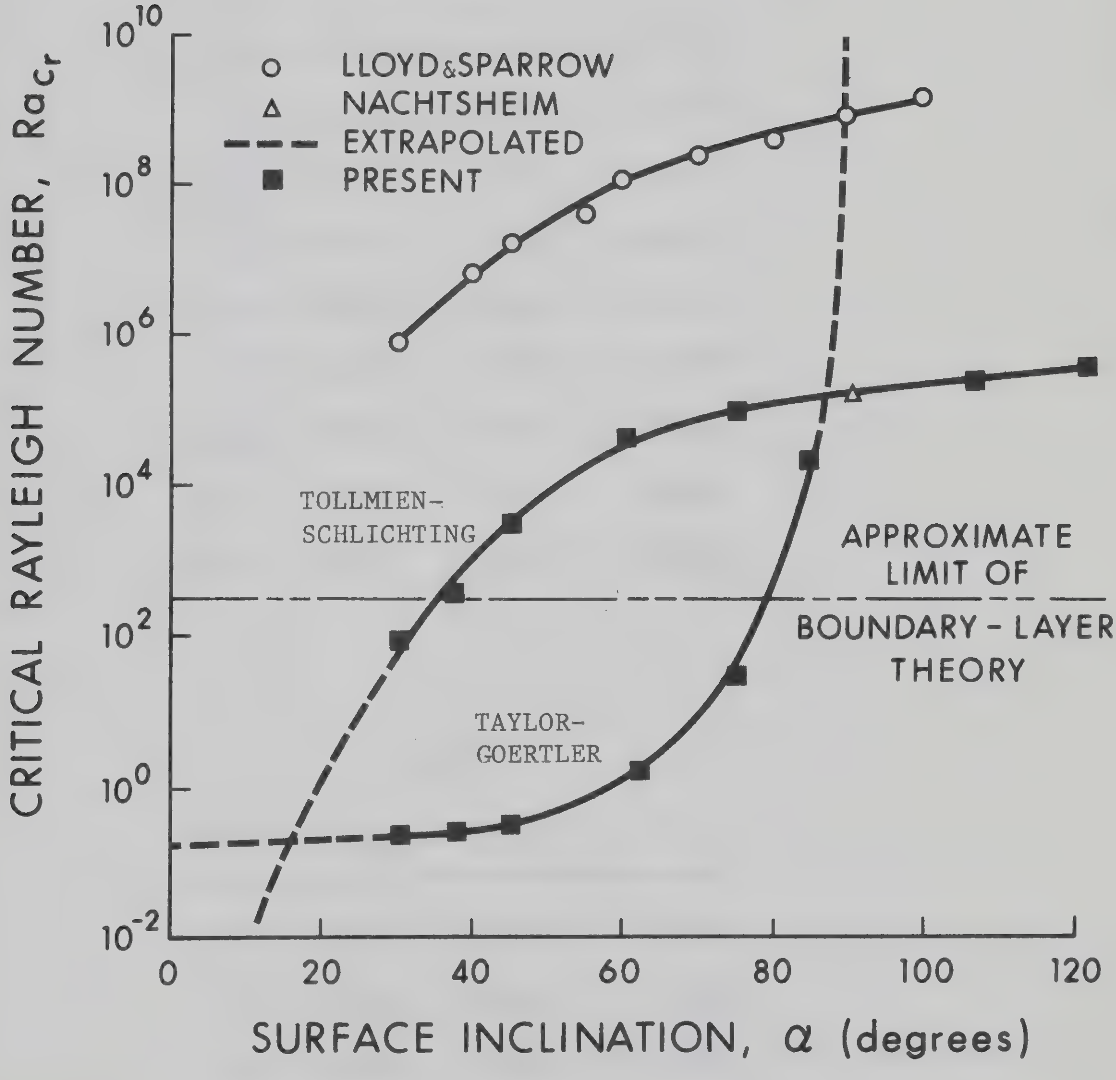


FIGURE 14. COMPARISON OF CRITICAL RAYLEIGH NUMBERS

TABLE 1

EFFECT OF SURFACE INCLINATIONS ON MEAN-FLOW EIGENVALUES

$\xi = 0.2$

α	$f_{\zeta\zeta}$	χ_ζ	π
0	2.2420	-0.5664	-1.3672
30	4.9133	-0.7273	-1.0211
45	4.7693	-0.7194	-0.9461
60	3.8988	-0.6665	-0.8643
75	2.7104	-0.5733	-0.6670
90	1.6898	-0.4818	0.0000
105	2.6030	-0.5739	0.6642
120	3.5569	-0.6588	0.8698

$\xi = 1.0$

α	$f_{\zeta\zeta}$	χ_ζ	π
0	2.2420	-0.5664	-1.3672
30	8.7905	-0.8871	-0.7546
45	8.0485	-0.8542	-0.6226
60	5.8518	-0.7532	-0.4786
75	3.4578	-0.6207	-0.2830
90	1.6898	-0.4818	0.0000
105	3.5170	-0.6107	0.2872
120	5.8587	-0.7422	0.4844

TABLE 2

VARIATION WITH POSITION ALONG THE SURFACE
OF MEAN-FLOW EIGENVALUES

$\alpha = 45°$

ξ	$f_{\zeta\zeta}$	χ_ζ	π
0.00000	2.0762	−0.5543	−4.0101
0.00625	1.2073	−0.4792	−2.3551
0.01875	2.4157	−0.5717	−1.7460
0.04375	2.9287	−0.6083	−1.4169
0.10000	3.7886	−0.6632	−1.1420
0.20000	4.7693	−0.7194	−0.9461
0.40000	5.9928	−0.7790	−0.7857
0.60000	6.7987	−0.8108	−0.7100
1.00000	8.0485	−0.8542	−0.6226

$\alpha = 75°$

ξ	$f_{\zeta\zeta}$	χ_ζ	π
0.00000	0.8112	−0.4146	−11.6288
0.00625	0.7013	−0.4010	−4.9209
0.01875	1.8397	−0.5270	−2.2746
0.02500	1.8995	−0.5316	−1.9479
0.05000	2.1095	−0.5411	−1.3775
0.10000	2.3898	−0.5549	−0.9645
0.20000	2.7104	−0.5733	−0.6670
0.40000	3.0389	−0.5936	−0.4601
0.60000	3.2144	−0.6049	−0.3713
1.00000	3.4578	−0.6207	−0.2830

TABLE 3

COMPARISON OF THE SURFACE PRESSURE DEPARTURES

$\underline{\xi = 0.2}$

$$\frac{P_d L^2 \sigma}{\rho \nu^2} \times 10^4$$

α	PRESENT	KIERKUS
30	-3.4306	-3.4502
45	-2.5862	-2.5834
60	-1.7502	-1.7370
75	-0.8985	-0.8750

$\underline{\xi = 1.0}$

$$\frac{P_d L^2 \sigma}{\rho \nu^2} \times 10^4$$

α	PRESENT	KIERKUS
30	-5.2694	-5.1595
45	-3.9702	-3.8633
60	-2.6896	-2.5976
75	-1.3530	-1.3084

TABLE 4

EIGENVALUES FOR THE TOLLMIEN-SCHLICHTING WAVE DISTURBANCES

Ra_X	$\frac{Ra_X^{\frac{1}{5}}(1+\omega^2)}{\xi}\lambda_r$ UPPER BRANCH	LOWER BRANCH	$\frac{Ra_X^{\frac{1}{5}}(1+\omega^2)}{\xi^2}B_r$ UPPER BRANCH	LOWER BRANCH
$\alpha = 120°$				
1.000×10^6	5.2313	1.6999	2028.3	901.4
4.655×10^5	3.8569	2.9651	1515.7	1253.3
$\alpha = 105°$				
1.000×10^6	5.9366	1.4709	2303.4	845.2
3.430×10^5	5.4234	2.6036	1930.9	1149.8
2.383×10^5		3.9108		1478.2
$\alpha = 90°$				
4.000×10^6		0.8939		723.1
2.800×10^6	12.7814		3956.4	
1.900×10^6	7.4294		2615.7	
1.000×10^6	7.0256	1.5044	2437.8	832.6
1.720×10^5	4.8850	4.6751	1587.1	1540.2
$\alpha = 75°$				
1.000×10^6	10.7727	1.3279	2951.4	744.2
7.290×10^5	10.9724		2917.9	
5.120×10^5	8.8801	1.7643	2517.2	856.3
1.25×10^5	7.4319	3.7765	1970.1	1263.2

TABLE 4 (CONTINUED)

α = 60°				
1.000×10^6	8.1416	1.2275	2083.3	657.5
5.120×10^5	10.0022		2408.3	
2.160×10^5	12.6150	2.3081	2689.2	881.4
6.400×10^4	10.4028	4.3460	2062.1	1188.3
α = 45°				
1.000×10^6	9.5018	1.1528	1528.7	553.3
2.160×10^5	15.2573	2.0586	2370.5	709.9
9.112×10^4	18.5805		2894.0	
8.000×10^3	15.9489	7.8290	1797.5	1105.1
α = 37.5°				
1.000×10^6	9.5015	1.0921	1393.6	484.8
2.160×10^5	15.0876	1.8732	2146.5	601.3
6.400×10^4	20.3577	2.8793	2916.4	698.5
8.000×10^3	20.9067	5.4852	2155.6	785.5
1.000×10^3	18.4608	10.1014	1301.7	813.4
5.12×10^2	16.8832	12.2704	1059.4	824.0
α = 30°				
1.000×10^6	9.4161	1.0081	1280.0	404.2
2.160×10^5	15.0306		2009.3	
6.400×10^4	20.8175	2.4040	2756.3	535.0
4.288×10^4	22.6427		2974.3	
8.000×10^3	24.5873	4.1110	2419.9	557.3
1.000×10^3	23.7006	6.5226	1570.8	519.4
91.12	16.1803	12.5260	679.6	529.4

TABLE 5

EIGENVALUES FOR THE TAYLOR-GOERTLER ROLL-VORTEX DISTURBANCES

Ra_X	$\frac{\Omega_r}{\xi} Ra_X^{\frac{1}{5}}(1+\omega^2)$					
	$\alpha = 85°$	$\alpha = 75°$	$\alpha = 60°$	$\alpha = 45°$	$\alpha = 37.5°$	$\alpha = 30°$
1.000×10^6	12.920	23.896	32.098			
5.120×10^5	11.970	23.301				
2.160×10^5	10.277	22.415	30.960	36.186	38.176	
6.400×10^4	6.668	20.933	29.725	35.246	37.341	39.087
2.700×10^4	1.915					
8.000×10^3		17.784	26.964	33.103	35.457	37.426
1.000×10^3		13.484	23.064	30.063	32.645	34.795
1.250×10^2		6.956				
83.74			16.459	24.533	27.275	29.524
30.52			13.315	21.429	24.148	26.379
6.592				16.059	18.343	20.291
1.953				10.778	12.425	13.935